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Optimal Control of an Inhomogeneous Heat Problem by Using Measure Theory

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Abstract

In this paper we consider an optimal control problem for an inhomogeneous Heat equation. We transfer the problem into a moment problem. Then this moment problem is changed to measure theoretic control problem, and the new problem is converted to an infinite dimensional linear programming problem. Finally we approximate the infinite dimensional linear programming problem to a finite dimensional one and the solution to this problem is used to find a piecewise constant control for the original problem.

Keywords: *Heat equation, measure theory, semigroup theory, moment problem, optimal control, linear programming.*

1. Statement of the problem

Let us consider the following problem:

1.1 $$\frac{\partial P(x,t)}{\partial t} = c^2 \nabla^2 P(x,t) + b(x)u(t)$$

$x \in Q^0, 0 \le t \le T$

1.2 $P(x,0) = F(x)$ $\qquad x \in Q$

1.3 $P(x,t) = 0$ $\qquad x \in \partial Q, 0 \le t \le T$

where Q is a n-cell in n-dimensional Euclidean space R^n (for n=1,2), with interior Q° and boundary ∂Q is the boundary of Q, u(.) is a scalar valued control function, ∇^2 is the Laplacian operator, c^2 is a constant, F(.) is a measurable function, and b(.) is continuous on Q.

The control function u(.) will be admissible if it is a measurable function on J=[0,T] and,

1: It takes values in the set [-1,1] for t$\in$ [0,T],
2: The solution of the system (1.1)-(1.3) corresponding to this control function, that is $P_u(x,t)$, satisfies the terminal condition :

1.4 $$P_u(x,T) = G(x) \qquad x \in Q$$

where $G(\cdot) \in L_2(Q)$ is the desired final state We assume that the set of all admissible controls is nonempty and denote it by U.
Our optimal control problem consists of finding a control $u(\cdot) \in U$ which minimizes the functional:

1.5 $$J(u) = \int_0^T f0(t, u(t))dt,$$

where $f0 \in C(\Omega), C(\Omega)$ is the space of all continuous functions on $\Omega = [0,T] \times [-1,1]$ with the uniform topology. In the following we replace the above problem with another one in which we introduce an approximate piecewise constant optimal control by using measure theory.

A boundary controllability theory for hyperbolic and parabolic partial differential equations has been studied and some results have been obtained (Fattorini & Russell,1971; Kamyad,1992). Some authors used measure theory to solve boundary optimal control problem for the diffusion equation (Kamyad *et al*., 1992). Also, in 1996, Farahi *et al*., solved a boundary optimal control problem of a homogenous linear wave equation by using measure theory (Farahi *et al*., 1996a,1996b). Recently Alavi *et al*., (Alavi *et al*., 1998) used measure theory and found an optimal control of inhomogeneous wave problem with internal control.

2. Obtaining moment problem

Let $H^1(Q)$ denote the usual Sobolev space on Q, i.e.,
$H^1(Q) = \{f : f \text{ and } f' \in L_2(Q)\}$,
and define

$$V = \{f \in H^1(Q) : f(x) = 0 \quad for\ x \in \partial Q\}.$$

Also, let $y(t)=P(\cdot,t)$, then we may write equations (1.1)-(1.3) as:

2.1 $$y'(t)=Ay(t)+bu(t) \qquad t\in J,\, y\in V,$$

2.2 $$y(0)=y_0.$$

where $y_0=F(x)$ and A is a sectorial operator defined by $A\phi=c^2\nabla^2\phi$ with domain V. Then we may write the solution of (2.1)-(2.2) as

2.3 $$y(t)=S(t)y_0+\int_0^t S(t-\tau)bu(\tau)d\tau \qquad t\in J,$$

where S(t) is a semigroup generated by the operator A (Banks,1983).

Now let the eigenvalues and eigenfunctions of the operator A be given as follows:

$$Ae_n(x)=-\lambda_n e_n(x),$$

$$e_n(x)=0, \qquad x\in\partial Q, \qquad n=1,2,\ldots;$$

Let the expansion of $h(x)\in L_2(Q)$ in terms of eigenfunction be $h(x)=\sum_{n=1}^{\infty}h_n e_n(x)$, then we can write the semigroup *S(t)* as:

$$S(t)h=\sum_{n=1}^{\infty}h_n e^{-\lambda_n t}e_n(x).$$

Therefore, by (2.3) the solution of (2.1)-(2.2) is of the following form:

2.4 $$y(t)=\sum_{n=1}^{\infty}\left[F_n e^{-\lambda_n t}+b_n\int_0^t e^{-\lambda_n(t-\tau)}u(\tau)d\tau\right]e_n(x),$$

where F_n and b_n are respectively the Fourier coefficients of $F(x)$ and $b(x)$.

Since $P(x,t)=y(t)$, so by (2.4) the solution of the system (1.1)-(1.3) can be written as:

2.5 $$P(x,t)=\sum_{n=1}^{\infty}\left[F_n e^{-\lambda_n t}+b_n\int_0^t e^{-\lambda_n(t-\tau)}u(\tau)d\tau\right]e_n(x)$$

Let the expansion of $G(x)$, in terms of eigenfunctions be

$$G(x)=\sum_{n=1}^{\infty}G_n e_n(x);$$

By (1.4) and (2.5) we must find a control function u such that satisfies the following conditions:

$$F_n e^{-\lambda_n T} + b_n \int_0^T e^{-\lambda_n (T-t)} u(t)dt = G_n \, . \qquad n = 1,2,\ldots$$

Assume in the Fourier series, $b(x) = \sum b_n e_n(x), b_n \neq 0$, then the above relation can be written as:

$$\text{2.6} \qquad \int_0^T e^{-\lambda_n (T-t)} u(t)dt = \frac{1}{b_n}\left(G_n - F_n e^{-\lambda_n T}\right).$$

$n = 1,2,\ldots.$

Now let

$$\text{2.7} \qquad \varphi_n(t,u) = e^{-\lambda_n (T-t)} u(t), \quad a_n = \frac{1}{b_n}\left(G_n - F_n e^{-\lambda_n T}\right), \; n = 1,2,\ldots$$

Hence, we must find a control function $u(\cdot): J \to [-1,1]$ such that:

$$\text{2.8} \qquad \int_0^t \varphi_n(t,u)dt = a_n \qquad n = 1,2,\ldots,$$

and minimizes the functional (1.5). We call this problem an optimal moment problem, and consider it in the next section.

3. Modified optimal moment problem

Now we replace the above moment minimization problem with another one as follow:

1: For a fixed control function $u(\cdot) \in U$, the mapping

$$\text{3.1} \qquad \Lambda_u : F \mapsto \int_0^t F(t,u)dt, \qquad \forall F \in C(\Omega),$$

defines a positive linear functional on $C(\Omega)$.

2: By the Riesz representation theorem, there exists a unique positive Radon measure μ_u on Ω such that

$$\text{3.2} \qquad \Lambda_u(F) = \int_0^t F d\mu_u = \mu_u(F) \qquad \forall F \in C(\Omega).$$

This measure ?‡$_u$ is required to have certain properties which are abstracted from the definition of admissible controls. First by (3.2)

$$|?‡_u(F)| \le T \sup_{\Omega} |F(t,u)|,$$

hence

$$?a_u(1) \le T \ .$$

Next, by (2.8) we have

$$?‡_u(?g_n) = a_n \qquad\qquad n = 1,2,....$$

Finally, consider functions $H(\cdot) \in C(\Omega)$ which do not depend on *u*, we have

$$\int_{\Omega} Hd?‡ = \int_0^T H(t,u(t))dt = a_H,$$

where a_H is the Lebesgue integral of H. let $M^+(\Omega)$ be the set of positive Radon measures on Ω. The set Q_0 is defined as a subset of $M^+(\Omega)$ such that :

$$Q_0 = S_1 \text{ I } S_2 \text{ I } S_3$$

where,

$$S_1 = \{? \in M^+(\Omega) : ?‡(1) \le T \},$$
$$S_2 = \{?ê \in M^+(\Omega) : ?ê(?‡_n) = a_n, n = 1,2,... \},$$
$$S_3 = \{?º \in M^+(\Omega) : ?‡(H) = a_H, H \in C(\Omega)$$

and H is independent of $u\}$.

So the new optimization problem consists of minimizing the linear functional
$I : Q_0 \to R$ defined by

$$I(?†) = \int_{\Omega} f_0 d?†$$

over the set Q_0 .

Proposition 3.1
The measure-theoretical control problem, which consist of finding the minimum of the functional I over the set Q_0, attains its minimum, say ?‡*, in Q_0 (Rubio, 1986).

4. Approximation of the optimal control by a piecewise constant control

Corresponding to each piecewise constant admissible control $u(\cdot)$, we may associate a measure ?‡$_u$, in $M^+(\Omega)$ I S_1 I S_2. Let Q_1 be the set of all such measures ?‡$_u$. When the space $M^+(\Omega)$, has the week*-topology, Q_1 is dense in $M^+(\Omega)$ I S_1 I S_2 (Theorem 1 of Ghouila-Houri, 1967). A basis of closed neighborhood in the week*- topology is given by sets of the form

4.1 $$\{?;|?‡(H_n) \leq \in, n = 1,2,\ldots,k+1| \},$$

where k is an integer $H_n \in C(\Omega, n = 1,2,\ldots,k+1)$, and ?‡ ≥ 0. In any week*- neighborhood of ?‡* (the minimizing measure), we can find a measure ?‡$_u$, corresponding to a piecewise control function $u(\cdot)$. In particular we can put

$$H_1 = f_0, H_2 = ?9_1, H_3 = ?9_2, \ldots, H_{k+1} = ?9_k;$$

then we can find a piecewise constant control $u_k(\cdot)$, such that

$$\left| \int_0^T f_0(t,u_k)dt - ?‡^*(f_0) \leq ?‡ \right|$$

4.2 $$\left| \int_0^T ?Ó_n(t,u_k)dt - a_n \right| \leq ?‡ \qquad n = 1,2,\ldots,k.$$

Therefore, by using the piecewise constant control $u_k(\cdot)$, we can reach within ?‡ of the minimum value ?â*(f_0).

Now, we analyze the relation between the desired final state $G(\cdot)$ and $P_k(\cdot,T)$ for the one-dimension state, $P_k(\cdot,T)$ is the final state attained by using the control $u_k(t)$). Let $Q = [0, L]$, where L is a fixed positive real number; in this case, the eigenfunctions and the corresponding eigenvalues of the operator $A = c^2 \frac{\partial^2}{\partial x^2}$ are as

4.3 $$e_n = \sin\left(\frac{n?‡}{L}x\right) \qquad x \in Q$$

4.4 $$?3_n = \left(\frac{cn?‡}{L}\right)^2 \qquad n = 1,2,...$$

Now we can show that if ?‡is chosen small enough, and k large enough, the distance between $G(\cdot)$ and $P_k(\cdot, T)$ in $L_2(Q)$ can be made as small as desired.

Proposition 4.1

Given $? \geq 0$, we may choose k and ?‡such that

4.5 $$\int_0^L [P_k(x,T) - G(x)]^2 dx \leq ?ª$$

Proof

Without loss of generality, we assume that $c = 1, T = L = 1$. Thus by (2.5) and (4.3)-(4.4) we have

$$P_k(x,1) = \sum_{n=1}^{\infty}\left[F_n \exp\left(-n^2?š^2\right) + b_n \int_0^1 \exp\left(-n^2?š^2(1-t)\right)u_k(t)dt\right]\sin(n?š x)$$

.

Let

$$?_n = F_n \exp\left(-n^2?^2\right) + b_n \int_0^1 \exp\left(-n^2?‡^2(1-t)\right)u_k(t)dt, \quad n = 1,2,...,$$

the Fourier coefficients $?‡_n$ of $P_k(\cdot,1)$, satisfy

4.6 $$|?a_n| \leq |F_n| \exp\left(-n^2?à^2\right) + |b_n| \left|\int_0^1 \exp\left(-n^2?‡^2(1-t)\right)dt\right| \quad n = 1,2,...$$

$$\leq \frac{|F_n|}{n^2?ï^2} + \frac{|b_n|}{n^2?‡^2}.$$

Since F_n and b_n are respectively, the Fourier coefficients of $F(\cdot)$ and $b(\cdot)$, then for the same integer k_1, when $n \geq k_1$, we have $|F_n| \leq 1$ and $|b_n| \leq 1$. Thus

$$|?_n| \leq \frac{2}{n^2 ?^2}.$$

Also, since the desired final state $G(x) = \sum G_n \sin(n ? x)$ is reachable by an admissible control, $|G_n|$ satisfies the same inequality as $|?_n|$. Thus,

$$\int_0^L [P_k(x,T) - G(x)]^2 dx = \frac{1}{2}\sum_{n=1}^{k}(?_n - G_n)^2 + \frac{1}{2}\sum_{n=k+1}^{\infty}(b_n - G_n)^2 ,$$

where for $k \geq k_1$,

$$\sum_{n=k+1}^{\infty}(?_n - G_n)^2 \leq \frac{16}{?^4}\sum_{n=k+1}^{\infty}\frac{1}{n^4}$$

Since the last summation in this expression is the tail of a convergent series we may choose k such that $k \geq k_1$ and

4.7 $$\sum_{n=k+1}^{\infty}(?_n - G_n)^2 \leq \frac{?}{2}.$$

Also, we choose $? = \frac{1}{2b_0}\sqrt{\frac{?}{k}}$, where $b_0 = 2\sup_{x\in[0,1]}|b(x)|$. In the neighborhood defined by choosing ? and k as above, there exists a $?_u$ corresponding to a piecewise constant control $u_k(\cdot)$ for which we have (4.2). Thus by (2.7) we can write

$$?_{n=1}^{k}(?_n - G_n)^2 ? ?_{n=1}^{k} ? exp(?n^2 ?) ? b_n \int_0^1 exp(?n^2 ?(1 ?))u_k(t)dt ? G_n ?^2$$

$$? ?_{n=1}^{k} b_n^2 (\int_0^1 ?(t,u_k)dt ? a_n)^2 ? b_0^2 ?_{n=1}^{k}\left|\int_0^1 ?(t,u_k)dt ? a_n\right|^2$$

by (4.2) we have

4.8 $$\sum_{n=1}^{k}(?_n - G_n)^2 \leq b_0^2 k ?^2 \leq \frac{?}{2}.$$

Therefore by (4.7)-(4.8) we have

$$\int_0^1 \left[P_k(x,T)-F(x)\right]^2 dx = \sum_{n=1}^{k}\left(?\rho_n - F_n\right)^2 + \sum_{n=k+1}^{\infty}\left(?\rho_n - F_n\right)^2 \le \frac{?^{..}}{2} + \frac{?^{..}}{2} = ?o$$

5. Approximation of the optimal measure

Now we develop a method for the estimation of a nearly-optimal piecewise constant control. In this method we follow Kamyad *et al.*, (1991). First we obtain an approximate value of the optimal measure ?‡. Let $Q(M_1, M_2)$ be the set of measures in $M^+(\Omega)$ satisfying

$$\text{5.1} \qquad \begin{cases} ?(1) \le T \\ ?(?_n) = a_n, n = 1,2,...,M_1 \\ ?‡(H_n) = a_{H_n}, n = 1,2,...,M_2 \end{cases}$$

Define $?‡_{Q(M_1,M_2)}(f_0)$ as

$$?¾_{Q(M_1,M_2)}(f_0) = \min\{?¾(f_0) : ?‡ \in Q(M_1, M_2)\},$$

then

$$\lim_{M_1,M_2\to\infty} ?^{*}_{Q(M_1,M_2)}(f_0) = ?^{*}(f_0)$$

(see Proposition 3.1 in Kamyad *et al.*, 1992).
The set $\Omega = [0,T]\times[-1,1]$ will be covered with a grid, by taking $m_1 + 1$ and $m_2 + 1$ points along the t-axis and u-axis, respectively. These points will be equidistant, at distances $\frac{T}{m_1}$ and $\frac{2}{m_2}$, each separately in the order mentioned. Now Ω is divided to $N = m_1.m_2$ equal rectangles Ω_j, $j = 1,2,...,N$, we choose points $Z_j \in \Omega_j$ and let

$$?= \{Z_j ; j = 1,2,...,N\}.$$

Now let $P(M_1, M_2, ?) \subseteq R^N$ be the set of all $(?_1, ?_2, ..., ?_N)$ defined by:

$$\begin{cases} ?\tilde{Q}_j \geq 0,\ j = 1,2,\ldots,N \\ \left|\sum_{j=1}^{N} ?_j ?_i(Z_j) - a_i\right| \leq ?,\ i = 1,2,\ldots,M_1 \\ \left|\sum_{j=1}^{N} ?_j H_i(Z_j) - a_{H_i}\right| \leq ?,\ i = 1,2,\ldots,M_2 \end{cases}$$

we can write the following proposition:

Proposition 5.1

For every $? > 0$, the problem of minimizing the function $\sum_{j=1}^{N} ?_j f_0(Z_j), Z_j \in ?$, on the set $P(M_1, M_2, ?)$, has a solution for $N = N(?)$ sufficiently large. the solution satisfies

$$?_{Q(M_1,M_2)}(f_0) + ?(?) \leq \sum_{j=1}^{N} ?_j f_0(Z_j) \leq ?_{Q(M_1,M_2)}(f_0) + ?$$

where $?(?)$ tends to zero as $?$ tends to zero.

Proof

The proof is the same as that of the Theorem III (Rubio1986). Thus we can compute $?_{Q(M_1,M_2)}(f_0)$ (the approximate value of $?(f_0)$), where in fact $?_{Q(M_1,M_2)}(f_0)$ is the approximate solution of the following linear programming problem:

Minimize

$$\sum_{j=1}^{N} ?_j f_0(Z_j)$$

subject to:

5.2

$$\begin{cases} ?_j ? 0,\ j ? 1,2,\ldots,N \\ \sum_{j?1}^{N} ?_j ?_i(Z_j) ? a_j,\ i ? 1,2,\ldots,M_1 \\ \sum_{j?1}^{N} ?_j H_i(Z_j) ? a_{H_j},\ i ? 1,2,\ldots,M_2 \\ ?_{Q(M_1,M_2)}(1) ? \sum_{j?1}^{N} ?_j 1(Z_j) ? \sum_{j?1}^{N} ?_j ? T. \end{cases}$$

or

Minimize

5.3 $$\sum_{j=1}^{N+1} ?Ø_j f_0(Z_j)$$

over the set of coefficient $?‡_j \geq 0,\ j = 1,2,\ldots,N+1$, such that

5.4 $$\begin{cases} \sum_{j=1}^{N+1} ?‡_j ?_i(Z_j) = a_j,\ i = 1,2,\ldots,M_1 \\ \sum_{j=1}^{N+1} ?‡_j H_i(Z_j) = a_{H_j},\ i = 1,2,\ldots,M_2 \\ \sum_{j=1}^{N+1} ?‡_j = T \end{cases}$$

We used one slack variable $?‡_{N+1}$, to put the last inequality in (5.2) in the equality form.

Remark

If $\sum_{j=1}^{N} ?‡_j = T$ then $?‡_{N+1} = 0$. Also we choose $H_i(t,u), i = 1,2,\ldots,M_2$ as follows:

$$H_i(t,u) = \begin{cases} 1, t \in J_i \\ 0, otherwise \end{cases}$$

where $J_i = \left[\frac{(i-1)T}{m_1}, \frac{iT}{m_1} \right], i = 1,2,\ldots,m_1$, so:

$$a_{H_i} = \int_0^T H_i(t,u)dt = \frac{T}{m_1}.$$

Now by using the solution of the finite dimensional linear programming (5.3)-(5.4), we can construct an approximate control function. Let $Z_n = (t_n, u_n), n = 1,2,\ldots,N$ to each n we can attribute a pair (i,j) as follows :

$$n = m_1(j-1) + i \qquad 1 \leq i \leq m_1, \quad 1 \leq j \leq m_2,$$

assume $K_{ij} = ?þ_n$. We define a piecewise-constant control as:

5.5 $$u(t) = u_n \qquad t \in B_{ij}$$

where

5.6 $$B_{ij} = \left[t_{i-1} + \sum_{l=1} K_{ij},\ t_{i-1} + \sum_{l=1} K_{ij} \right]$$

Since those intervals B_{ij} for which $K_{ij} = 0$ are reduced to a point, they do not contribute anything to intervals and so can be ignored. By using the piecewise-constant control we can compute the final state $P(x,T)$.

Example 5.1

Consider the heat equation with internal control

5.7 $P_t(x,t) = P_{xx}(x,t) + xu(t) \qquad (x,t) \in \Omega = (0,1) \times (0,0.4)$

with the following initial and boundary conditions:

5.8 $$\begin{cases} P(x,0) = 10(x - x^2), 0 \le x \le 1 \\ P(0,t) = P(1,t) = 0, 0 \le t \le 0.4 \end{cases}$$

We are going to construct the optimal control function $u(\cdot):[0,0.4] \to [-1,1]$, such that the solution of the system (5.7)-(5.8) corresponding to this control function satisfies the following desired final condition:

$$P(x,0.4) = 0 \qquad x \in [0,1],$$

and, minimizes the functional

5.9 $$J(u) = \int_0^2 |u(t)| dt .$$

We assume $M_1 = 10, M_2 = 20$, $m_1 = m_2 = 20$, so the set $\Omega = [0,0.4] \times [-1,1]$ is divided to $N = 400$ subrectangles. Also, we define $Z_i = (t_i, u_i), i = 1,2,\ldots,400$, as

$$t_{20i+1} = t_{20i+2} = \ldots = t_{20i+20} = 0.02i + 0.01 \qquad i = 0,1,2,\ldots,19$$

$$u_{1+i} = u_{21+i} = \ldots = u_{381+i} = \frac{2}{19} i - 1 \qquad i = 0,1,2,\ldots,19$$

So, the linear programming problem (5.3)-(5.4) changes to the following problem:
Minimize

$$\sum_{j=1}^{400} \Delta_j \left|u_j\right|$$

subject to:

$$\begin{cases} \Delta_j \geq 0 & j=1,2,\ldots,400 \\ \sum_{j=1}^{400} \Delta_j e^{-n^2\pi^2(0.4-t_j)} u_j = \begin{cases} \dfrac{20}{n^2\pi^2} e^{-0.4n^2\pi^2}, & n=2k-1 \\ 0 & n=2k \end{cases} & , n=1,2,\ldots,10 \\ \Delta_{20i+1}+\Delta_{20i+2}\ldots+\Delta_{20i+20}=0.02 & i=1,2,\ldots,19 \end{cases}$$

In this example, the cost function converges to the value 0.0903. The graph of the piecewise constant control function formed by using the above method, can be seen in Figure 5.1. The initial and final states are shown in Figure 5.2. We mention that $P(x,04)$ is approximated by only the first four terms of the series (2.5), that is:

$$P(x,04)=0.0430\sin(\pi x)-0.0028\sin(2\pi x)+0.0066\sin(3\pi x)-0.0025\sin(4\pi x)$$

Figure 5.1 – The nearly optimal control for Example 5.1.

Figure 5.2 – Initial and final actual states for Example 5.1.

6. Optimal control for the two-dimensional inhomogeneous heat equation

In this section let $Q=[0,L]\times[0,H]$ where L and H are fixed positive numbers; in this case, the eigenfunction and eigenvalues of the operator $A=c^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}\right)$ are as:

$$e_{mn}(x,y)=\frac{2}{\sqrt{LH}}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right), \qquad \lambda_{mn}=\left(\frac{m\pi}{L}\right)^2+\left(\frac{n\pi}{H}\right)^2 .$$

Therefore, we can write the solution of (1.1)-(1.3) as

6.1 $$P(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[b_{mn}e^{-\lambda_{mn}t}+F_{mn}\int_0^t e^{-\lambda_{mn}(t-\tau)}u(\tau)\,d\tau\right]e_{mn}(x)$$

Where b_{mn} and F_{mn} are respectively double Fourier sine coefficients of functions $b(x,y)$ and $F(x,y)$. So,by (2.6) we must find a control function $u(\cdot)$ such that:

$$\int_0^t e^{-\lambda_{mn}(T-t)}u(t)\,dt=\frac{1}{b_{mn}}\left(G_{mn}-F_{mn}e^{-\lambda_{mn}T}\right)$$

where G_{mn} is double Fourier sine coefficients of function $G(x,y)$. But, by the following correspondence

$$f : N \times N \to N$$
$$(m,n) \to 2^{m-1}(2n-1) = k\,,$$

the above relation can be written as the following from

$$\int_0^T e^{-?_k(T-t)}u(t)dt = \frac{1}{b_k}\left(G_k - F_k e^{-?_k T}\right) \qquad k = 1,2,\ldots$$

Therefore, by (2.7), we need to find a control function $u(\cdot):[0,T] \to [-1,1]$ such that satisfies in (2.8) and minimizes the functional (1.5). This problem is an optimal moment problem, and we considered it in the previous sections. We give an example in two-dimensional system.

Example 6.1

Consider the two-dimensional inhomogeneous heat equation

6.2 $\quad P_t(x,y,t) = P_{xx}(x,y,t) + P_{yy}(x,y,t) + xyu(t)$

for $(x,y,t) \in (0,?) \times (0,?) \times \left(0, \frac{3?}{2}\right)$, where the initial and boundary conditions are:

6.3 $\quad p(x.y.0) = 0.2\sin(x)\sin(y) \qquad (x,y) \in Q = [0,?] \times [0,?]$

6.4 $\quad P_t(x,y,0) = 0 \qquad (x,y) \in Q$

6.5 $\quad P(x,y,0) = 0 \qquad (x,y,t) \in \partial Q \times \left(0, \frac{3?}{2}\right].$

We are going to construct the optimal control function $u(\cdot):\left[0, \frac{3?}{2}\right] \to [-1,1]$, such that the solution of the system (6.2), (6.5) corresponding to this control function, satisfies the following desired final condition :

$$P\left(x, y, \frac{3?}{2}\right) = 0\,, \qquad (x,y) \in Q$$

and also, minimizes the functional

6.7 $$J(u)=\int_0^{\frac{3?í}{2}} u^2 dt .$$

Let $M_1 = 10, M_2 = 20, m_1 = m_2 = 20$, thus $N = 400$. Also, we select $Z_i = (t_i, u_i), i = 1,2,...,400$, as

$$t_{20i?‡} ?‡t_{20i?2} ?c..?ct_{20i?20} ?c\frac{3?¢}{40}(i ?c1), \qquad i = 0,1,2,...,19$$

$$u_{1+i} = u_{21+i} = ... = u_{381+i} = \frac{2}{19} i - 1, \qquad i = 0,1,2,...,19 .$$

So, the linear programming problem (5.3)-(5.4) changes to the following problem:

Minimize

$$\sum_{i=1}^{400} ?l_i u_i^2$$

subject, to,

$$\begin{cases} ?ø_j \geq 0 & j = 1,2,...,400 \\ \sum_{j=1}^{400} ?_j e^{-n\left(\frac{3?ú}{2}-t_j\right)} u_j = \begin{cases} -0.05e^{-3?‡} \\ 0 \end{cases}, & \begin{matrix} n = 2 \\ n = 5,8,10,13,17,18,20,25,26 \end{matrix} \\ ?Q_{20i+1} + ?Q_{20i+2} + ... + ?‡_{20i+20} = 0.1 & i = 1,2,...,19 \end{cases}$$

In this example the cost function takes the value of 0.0055. By using of the solutions of this finite dimensional linear programming problem and (5.5) we obtained an approximated piecewise constant control function. Figure 6.1 shows this control function. By (6.1) we can compute $P\left(x, y, \frac{3?‘}{2}\right)$, the initial and desired final states are shown in Figures 6.2-6.3. In Figure 6.3, $P\left(x, y, \frac{3?Þ}{2}\right)$ is approximated by only the first eleven terms of the series (6.1), that is:

$$P\left(x, y, \frac{3?}{2}\right) = 0.0121\sin(?x)\sin(?y) - 0.0060\sin(2?x)\sin(?y)$$
$$-0.0060\sin(?x)\sin(2?y) + 0.0029\sin(2?x)\sin(2?y) + 0.0037\sin(3?x)\sin(?y)$$
$$+0.0037\sin(?x)\sin(3?y) - 0.0017\sin(3?x)\sin(2?y) - 0.0017\sin(2?x)\sin(3?y)$$
$$-0.0023\sin(4?x)\sin(?y) - 0.0023\sin(?x)\sin(4?y) + 0.0010\sin(3?x)\sin(3?y)$$

Figure 6.1- The nearly optimal control for Example 6.1.

Figure 6.2- Initial state for Example 6.1.

Figure 6.3- Final state actually achived for Example 6.1.

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