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A Note on Transformation Semigroups

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Abstract

In this note we study the transformation semigroup (X,S), where S is a finite union of its subsemigroups.

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Preliminaries:

By a transformation semigroup $(X,S,?)$ (or simply (X,S)) we mean a compact Hausdorff topological space X, a discrete topological semigroup S with identity e and a continuous map ?: $X\times S\to X$ (?$(x, s) = xs$ $(\forall x\in X, \forall s\in S)$) such that:

- $\forall x\in X \quad xe = x,$
- $\forall x\in X \quad \forall s, t\in S \quad x(st) = (xs)t.$

In the transformation semigroup (X,S) we have the following definitions:

1. For each $s\in S$, define the continuous map ?: $X\to X$ *by* x?$=xs$ $(\forall x\in X)$, we used to write s instead of $?^s$. The closure of $\{?^s \mid s\in S\}$ in X^X with pointwise convergence, is called the enveloping sermigroup (or Ellis semigroup) of (X,S) and it is written by $E(X,S)$ or simply $E(X)$. $E(X,S)$ has a semigroup structure (Ellis, 1969, Chapter 3), a nonempty subset K of $E(X,S)$ is called a right ideal if $KE(X,S)\subseteq K$, and it is called a minimal right ideal if none of the right ideals of $E(X,S)$ is a proper subset of K.

2. A nonempty subset Z of X is called invariant if $ZS\subseteq Z$, moreover it is called minimal if it is closed and none of the closed invariant subsets

of X is a proper subset of Z. Let $a\in X$, A be a nonempty subset of X and C be a nonempty subset of $\mathrm{E}(X,S)$, we introduce the following sets:

$\mathrm{F}(a,C)=\{p\in C \mid ap=p\}$, $\mathrm{F}(A,C)=\{p\in C \mid \forall b\in A \quad bp=b\}$,

$\overline{\mathrm{F}}(A,C)=\{p\in C \mid Ap=A\}$, $\mathrm{J}(C)=\{p\in C \mid p^2=p\}$.

3. Let $a\in X$, A be a nonempty subset of X and K be a closed right ideal of $\mathrm{E}(X,S)$, then (Sabbaghan and Shirazi, 2001a, Definition 1):

• We say K is an a-minimal set if:
- $aK= a\mathrm{E}(X,S)$,
- K dose not have any proper subset like L, such that L is a closed right ideal of $\mathrm{E}(X,S)$ with $aL = a\mathrm{E}(X,S)$.

• We say K is an $A\text{-}\overline{\text{minimal}}$ set if:
- $\forall b\in A$ $bK= b\mathrm{E}(X,S)$,
- K dose not have any proper subset like L, such that L is a closed right ideal of $\mathrm{E}(X,S)$ with $bL = b\mathrm{E}(X,S)$ for all $b\in A$.

• We say K is an $A\text{-}\overline{\overline{\text{minimal}}}$ set if:
- $AK= A\mathrm{E}(X,S)$,
- K dose not have any proper subset like L, such that L is a closed right ideal of $\mathrm{E}(X,S)$ with $AL = A\mathrm{E}(X,S)$.

The sets of all a-minimal (resp. $A\text{-}\overline{\text{minimal}}$, $A\text{-}\overline{\overline{\text{minimal}}}$) sets is written by $\mathrm{M}_{(X,S)}(a)$ (resp. $\overline{\mathrm{M}}_{(X,S)}(A)$, $\overline{\overline{\mathrm{M}}}_{(X,S)}(A)$).

$\overline{\mathrm{M}}_{(X,S)}(A)$ and $\mathrm{M}_{(X,S)}(a)$ are nonempty ((Sabbaghan and Shirazi, 2001a, Theorem 2) and (Sabbaghan, *et al*., 1997, Proposition 3)).

4. Let A be a nonempty subset of X, we introduce the following sets (Sabbaghan and Shirazi, 2001b, Definition 1):

$\mathrm{P}(X,S) = \{(x,y)\in X\times X \mid \exists p\in \mathrm{E}(X,S) \quad xp=yp\}$,

$\mathrm{P}_A(X,S) = \{(x,y)\in X\times X \mid \exists a\in A \quad \exists I\in \mathrm{M}_{(X,S)}(a) \quad \forall p\in I \quad xp=yp\}$,

$\overline{\mathrm{P}}_A(X,S) = \{(x,y)\in X\times X \mid \exists I\in \overline{\mathrm{M}}_{(X,S)}(A) \quad \forall p\in I \quad xp=yp\}$,

$\overline{\mathrm{M}}(X,S) = \{\varnothing\neq D\subseteq X \mid \forall K\in \overline{\mathrm{M}}_{(X,S)}(D) \quad \mathrm{J}(\mathrm{F}(D,K))\neq \varnothing\}$,

$\overline{\overline{\mathrm{M}}}(X,S) =\{\varnothing\neq D\subseteq X \mid \overline{\overline{\mathrm{M}}}_{(X,S)}(D)\neq\varnothing, \forall K\in \overline{\overline{\mathrm{M}}}_{(X,S)}(D) \quad \mathrm{J}(\overline{\mathrm{F}}(D,K))\neq \varnothing\}$,

5. Let (Y,S) be a transformation semigroup, a continuous map φ : $(X,S)\to (Y,S)$ is called a homomorphism if $\varphi(xs) = \varphi(x)s$ ($x\in X$, $s\in S$). Let φ: $(X,S)\to (Y,S)$ be an onto homomorphism, $\mathrm{R}(\varphi) = \{(x,y)\in X\times X \mid \varphi(x) = \varphi(y)\}$, $\Delta_X =\{(x,x) \mid x\in X\}$, A be a nonempty subset of X and B

be a nonempty subset of *Y*. We say (Sabbaghan and Shirazi, 2001b, Definition 7):

- (Y,S) is a distal (resp. A-distal, $A^{(\overline{M})}$distal) factor of (X,S) if $R(\varphi) \cap P(X,S) = \Delta_X$ (resp. $R(\varphi) \cap P_A(X,S) = \Delta_X$, $R(\varphi) \cap \overline{P}_A\ (X,S) = \Delta_X$),
- (X,S) is a distal (resp. B-distal, $B^{(\overline{M})}$distal) extension of (Y,S) if $R(\varphi) \cap P(X,S)=\Delta_X$ (resp. $R(\varphi) \cap P_{\varphi^{-1}(B)}(X,S) = \Delta_X$, $R(\varphi) \cap \overline{P}_{\varphi^{-1}(B)}(X,S) = \Delta_X$),
- (Y,S) is a proximal (resp. A-proximal, $A^{(\overline{M})}$proximal) factor of (X,S) if $R(\varphi) \subseteq P(X,S)$ (resp. $R(\varphi) \subseteq P_A(X,S)$, $R(\varphi) \subseteq \overline{P}_A\,(X,S)$),
- (X,S) is a proximal (resp. B-proximal, $B^{(\overline{M})}$proximal) extension of (Y,S) if $R(\varphi) \subseteq P(X,S)$ (resp. $R(\varphi) \subseteq P_{\varphi^{-1}(B)}(X,S)$, $R(\varphi) \subseteq \overline{P}_{\varphi^{-1}(B)}(X,S)$).

6. Let A be a nonempty subset of X, then (Sabbaghan and Shirazi, 2001a, Definition 13):

- (X,S) is distal if $E(X,S)$ is a minimal right ideal,
- (X, S) is called A- distal if for each $a\in A$, $E(X, S)\in M_{(X, S)}(a)$,
- (X, S) is called, $A^{(\overline{M})}$distal if $E(X, S)\in\ \overline{M}_{(X,S)}(A)$,
- (X, S) is called, $A^{(\overline{\overline{M}})}$distal if $E(X, S)\in\ \overline{\overline{M}}_{(X,S)}(A)$,

7. Let Z be a closed invariant subset of X, define:

$$h_{(X, S)}(Z) = \{n\in \mathbf{N} \cup \{0\} \mid \exists\, Z_0 , \ldots , Z_n \ni ((Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n) \wedge (\forall i\in \{0,\ldots,n\}\quad \forall j\in \{0,\ldots,n\}-\{i\}\quad Z_i \neq Z_j) \wedge (\forall i\in \{0,\ldots,n\}\quad Z_i \text{ is a closed invariant subset of } Z))\}.$$

Convention 1. In what follows (X, S) is a transformation semigroup, e is the identity of S and $S_0, S_1,\ldots, S_n$ are subemigroups of S, such that $e\in \bigcap_{i=0}^{n} S_i$ and $S = \bigcup_{i=1}^{n} S_i$.

Lemma 2.

1. $E(X,S)= \bigcup_{i=1}^{n} E(X, S_i)$.
2. $S= S_1\cdots S_n$ and $E(X,S) = E(X,S_1) \cdots E(X,S_n)$.

Proof.

1. If $p \in E(X,S)$, then there exists a net $\{s_\gamma\}_{\gamma\in\Gamma} \subseteq S$, such that $\lim_{\gamma\in\Gamma} s_\gamma = p$ (i.e., $\lim_{\gamma\in\Gamma} xs_\gamma = xp$ $(\forall x \in X)$), since $S = \bigcup_{i=1}^{n} S_i$, so there exists $i \in \{1, \ldots, n\}$ and a subnet $\{s_{\gamma_\lambda}\}_{\lambda\in\Lambda}$ of $\{s_\gamma\}_{\gamma\in\Gamma}$, such that $s_{\gamma_\lambda} \in S_i$, therefore $\lim_{\lambda\in\Lambda} s_{\gamma_\lambda} = p \in E(X, S_i)$. Thus $E(X,S) \subseteq \bigcup_{i=1}^{n} E(X,S_i)$.

2. Use $e \in \bigcap_{i=1}^{n} S_i$.

Theroem 3. Let A be a nonempty subset of X, then:

1. (X, S) is distal if and only if for each $i \in \{1, \ldots, n\}$, (X, S_i) is distal.
2. (X, S) is A-distal if and only if for each $i \in \{1, \ldots, n\}$, (X, S_i) is A-distal.
3. Let $A \in \bigcap_{i=1}^{n} \overline{\mathcal{M}}(X,S_i) \cap \overline{\mathcal{M}}(X,S)$. Then (X,S) is $A^{(\overline{\mathcal{M}})}$distal if and only if for each $i \in \{1, \ldots, n\}$, (X, S_i) is $A^{(\overline{\mathcal{M}})}$distal.
4. Let $A \in \bigcap_{i=1}^{n} \overline{\overline{\mathcal{M}}}(X,S_i) \cap \overline{\overline{\mathcal{M}}}(X,S)$. Then (X,S) is $A^{(\overline{\overline{\mathcal{M}}})}$distal if and only if for each $i \in \{1, \ldots, n\}$, (X, S_i) is $A^{(\overline{\overline{\mathcal{M}}})}$distal.

Proof.

1. (X,S) is distal if and only if (X,S) is X-distal (Sabbaghan and Shirazi, 2001a, Theorem 18), so this is a special case of (2).
2. We have (by (Sabbaghan and Shirazi, 2001a, Theorem 18)):

(X,S) is A-distal $\Leftrightarrow$ $\forall a \in A \quad J(F(a, E(X, S))) = \{e\}$

$$\Leftrightarrow \forall a \in A \quad J(F(a, \bigcup_{i=1}^{n} E(X, S_i))) = \{e\} \quad \text{(by Lemma 2)}$$

$$\Leftrightarrow \forall a \in A \quad \bigcup_{i=1}^{n} J(F(a, E(X, S_i))) = \{e\}$$

$$\Leftrightarrow \forall a \in A \quad \forall i \in \{1,\ldots,n\} \quad J(F(a, E(X, S_i))) = \{e\}$$

$\Leftrightarrow\ \forall i\in\{1,\ldots,n\}\quad (X, S_i)$ is A-distal .

3. We have (by (Sabbaghan and Shirazi, 2001a, Theorem 18)):

(X,S) is $A^{(\overline{\mathcal{M}})}$distal $\Leftrightarrow$ J (F(A, E(X, S))) = $\{e\}$

$\Leftrightarrow$ J (F(A, $\bigcup_{i=1}^{n}$E(X, S_i))) =$\{e\}$ (by Lemma 2)

$\Leftrightarrow$ $\bigcup_{i=1}^{n}$ J (F(A, E(X, S_i))) = $\{e\}$

$\Leftrightarrow\ \forall i\in\{1,\ldots,n\}$ J (F(A, E(X, S_i))) = $\{e\}$

$\Leftrightarrow\ \forall i\in\{1,\ldots,n\}$ (X, S_i) is $A^{(\overline{\mathcal{M}})}$distal.

4. Use a Similar method described in (3).

Theroem 4. Let n = 2 in Convention l and let A be a nonempty subset of X.

1. If (X,S) is distal then there exists $i\in\{1,2\}$ such that E(X,S) = E(X,S_i).
2. If (X,S) is A-distal, then for each $a\in A$ there exists $i\in\{1,2\}$ such that F(a, E(X,S)) = F(a, E(X,S_i)).
3. If $A\in\overline{\mathcal{M}}(X,S_1)\cap\overline{\mathcal{M}}(X,S_2)$ and (X,S) is $A^{(\overline{\mathcal{M}})}$distal, then there exists $i\in\{1, 2\}$ such that F(A, E(X,S)) = F(A, E(X,S_i)).
4. If $A\in\overline{\overline{\mathcal{M}}}(X,S_1)\cap\overline{\overline{\mathcal{M}}}(X,S_2)$ and (X,S) is $A^{(\overline{\overline{\mathcal{M}}})}$distal then there exists $i\in\{1, 2\}$ such that $\overline{\mathrm{F}}$(A, E(X,S)) = $\overline{\mathrm{F}}$(A, E(X,S_i)).

Proof.

1. By Theorem 3, (X,S_1), (X,S_2) are distal therefore E(X,S_1), E(X,S_2) and E(X,S) are groups. By Lemma 2 we have E(X,S) = E$(X,S_1)\cup$ E(X,S_2) Thus E$(X,S_1)\subseteq$ E(X,S_2) or E$(X,S_2)\subseteq$ E(X,S_1).

2. By Theorem 3, (X,S_1), (X,S_2) are A-distal therefore for each $a\in A$, F(a, E(X,S_1)), F(a, E(X,S_2)) and F(a, E(X,S)) are groups (Sabbaghan and Shirazi, 2001a, Theorem 18), moreover by Lemma 2, F(a, E(X,S)) =F(a, E(X,S_1)) $\cup$ F(a, E(X,S_2)), Thus F(a, E(X,S_1)) $\subseteq$ F(a, E(X,S_2)) or F(a, E(X,S_2)) $\subseteq$ F(a, E(X,S_1))

3. $F(A,\ E(X,S_1))$, $F(A,\ E(X,S_2))$ and $F(A,\ E(X,S))$ are groups (Sabbaghan and Shirazi, 2001a, Theorem 18), now use a similar method described in (2).
4. $\overline{F}(A,\ E(X,S_1))$, $\overline{F}(A,\ E(X,S_2))$ and $\overline{F}(A,\ E(X,S))$ are groups (Sabbaghan and Shirazi, 2001a, Theorem 18), now use a similar method described in (2).

Theorem 5. Let A be a nonempty subset of X, we have:

1. $P(X,S) = \bigcup_{i=1}^{n} P(X,S_i)$.
2. $P_A(X,S) = \bigcup_{i=1}^{n} P_A(X,S_i)$.
3. $\bigcup_{i=1}^{n} \overline{P}_A(X,S_i) \subseteq \overline{P}_A(X,S)$
4. If $A \in \bigcap_{i=1}^{n} \overline{M}(X,S_i) \cap \overline{M}(X,S)$. Then $\overline{P}(X,S) = \bigcup_{i=1}^{n} \overline{P}(X, S_i)$.

Proof. In all items we use Lemma 2 and (Sabbaghan and Shirazi, 2001b, Theorem 4).
1. $P(X,S) = P_X(X,S)$, so this is a special case of (2).

2. Let $x,y \in X$:
$(x,y) \in P_A(X, S)$

$$\Leftrightarrow \exists a \in A \quad \exists p \in F(a, E(X, S)) \quad xp=yp$$

$$\Leftrightarrow \exists a \in A \quad \exists p \in F(a, \bigcup_{i=1}^{n} E(X, S_i)) \quad xp=yp$$

$$\Leftrightarrow \exists a \in A \quad \exists p \in \bigcup_{i=1}^{n} F(a, E(X, S_i)) \quad xp=yp$$

$$\Leftrightarrow \exists a \in A \quad \exists i \in \{1,\dots,n\} \quad \exists p \in F(a, E(X, S_i)) \quad xp=yp$$

$$\Leftrightarrow \exists i \in \{1,\dots,n\} \quad (x,y) \in P_A(X, S_i)$$

$$\Leftrightarrow (x,y) \in \bigcup_{i=1}^{n} P_A(X, S_i).$$

Therefore $\mathrm{P}_A(X, S) = \bigcup_{i=1}^{n} \mathrm{P}_A(X, S_i)$.

3. For $i \in \{1,\ldots,n\}$, if $K \in \overline{\mathrm{M}}_{(X,S_i)}(A)$, then $\overline{K\mathrm{E}(X,S)}$ is a closed right ideal of $\mathrm{E}(X, S)$ and for each $a \in A$ we have $a\overline{K\mathrm{E}(X,S)} = a\mathrm{E}(X, S)$, thus there exists $L \in \overline{\mathrm{M}}_{(X,S)}(A)$ such that $L \subseteq \overline{K\mathrm{E}(X,S)}$ (Sabbaghan and Shirazi, 2001a, Corollary 3). Let $(x,y) \in X$, we have :

$$
\begin{aligned}
&(x,y) \in \bigcup_{i=1}^{n} \overline{\mathrm{P}}_A(X, S_i). \\
&\Rightarrow \exists i \in \{1,\ldots,n\} \quad (x,y) \in \overline{\mathrm{P}}_A(X, S_i) \\
&\Rightarrow \exists i \in \{1,\ldots,n\} \quad \exists K \in \overline{\mathrm{M}}_{(X,S_i)}(A) \quad \forall p \in K \quad xp=yp \\
&\Rightarrow \exists i \in \{1,\ldots,n\} \quad \exists K \in \overline{\mathrm{M}}_{(X,S_i)}(A) \quad \forall p \in K\,\mathrm{E}(X, S) \quad xp=yp \\
&\Rightarrow \exists i \in \{1,\ldots,n\} \quad \exists K \in \overline{\mathrm{M}}_{(X,S_i)}(A) \quad \forall p \in \overline{KE(X,S)} \quad xp=yp \\
&\Rightarrow \exists L \in \overline{\mathrm{M}}_{(X,S_i)}(A) \quad \forall p \in L \quad xp=yp \\
&\Rightarrow (x,y) \in \overline{\mathrm{P}}_A(X, S).
\end{aligned}
$$

Therefore $\bigcup_{i=1}^{n} \overline{\mathrm{P}}_A(X, S_i) \subseteq \overline{\mathrm{P}}_A(X, S)$.

4. Let $x,y \in X$, we have:

$$
\begin{aligned}
&(x,y) \in \overline{\mathrm{P}}_A(X, S) \\
&\quad \Leftrightarrow \exists p \in \mathrm{F}(A, \mathrm{E}(X, S)) \quad xp=yp \\
&\quad \Leftrightarrow \exists p \in \mathrm{F}(A, \bigcup_{i=1}^{n} \mathrm{E}(X, S_i)) \quad xp=yp \\
&\quad \Leftrightarrow \exists p \in \bigcup_{i=1}^{n} \mathrm{F}(A, \mathrm{E}(X, S)) \quad xp=yp \\
&\quad \Leftrightarrow \exists i \in \{1,\ldots,n\} \quad \exists p \in \mathrm{F}(A, \mathrm{E}(X, S_i)) \quad xp=yp \\
&\quad \Leftrightarrow \exists i \in \{1,\ldots,n\} \quad (x,y) \in \overline{\mathrm{P}}_A(X, S_i) \\
&\quad \Leftrightarrow (x,y) \in \bigcup_{i=1}^{n} \overline{\mathrm{P}}_A(X, S_i).
\end{aligned}
$$

Therefore $\overline{\mathrm{P}}_A(X, S) = \prod_{i=1}^{n} \overline{\mathrm{P}}_A(X, S_i)$.

Corollary 6. Let ?ì: $(X,S)\to(Y,S)$ be an onto homomorphisms, A be a nonempty subset of X and B be a nonempty subset of Y, then (with all the factors and extensions being under ?ì):

a. "(Y,S) is a distal factor of (X,S)" if and only if "for each $i\in\{1,\ldots,n\}$, (Y,S_i) is a distal factor of (X,S_i)".

b. "(Y,S) is an A-distal factor of (X,S)" if and only if "for each $i\in\{1,\ldots,n\}$, (Y,S_i) is an A-distal factor of (X,S_i)".

c. Let $A\in \bigcap_{i=1}^{n} \overline{\mathrm{M}}(X,S_i) \cap \overline{\mathrm{M}}(X,S)$, then "$(Y,S)$ is an $A^{(\overline{M})}$distal factor of (X,S)" if and only if "for each $i\in\{1,\ldots,n\}$, (Y,S_i) is an $A^{(\overline{M})}$distal factor of (X,S_i)".

d. "(X,S) is a distal extension of (Y,S)" if and only if "for each $i\in\{1,\ldots,n\}$, (X,S_i) is a distal extension of (Y,S_i)".

e. "(X,S) is a B-distal extension of (Y,S)" if and only if "for each $i\in\{1,\ldots,n\}$, (X,S_i) is a B-distal extension of (Y,S_i)".

f. Let $?^{-1}(B)\in \bigcap_{i=1}^{n} \overline{\mathrm{M}}(X,S_i) \cap \overline{\mathrm{M}}(X,S)$, then "$(X,S)$ is a $B^{(\overline{M})}$distal extension of (Y, S)" if and only if "for each $i\in\{1,\ldots,n\}$, (X, S_i) is a $B^{(\overline{M})}$distal extension of (Y, S_i)".

Proof. Use Theorem 5.

Note 7. Let $A_1, \ldots, A_n$ be nonempty subsets of X. We have:

1. If $\prod_{i=1}^{n} A_i \in \overline{\mathrm{M}}(X,S)$ and for each $j\in\{1,\ldots,n\}$, (X,S_j) is $A_j^{(\overline{M})}$distal, then (X,S) is $\prod_{i=1}^{n} A_i^{(\overline{M})}$distal.

2. If $\bigcup_{i=1}^{n} A_I \in \overline{\overline{\mathcal{M}}}\,(X,S)$ and for each $j \in \{1,\ldots,n\}$, (X,S_j) is $A_j^{(\overline{\overline{\mathcal{M}}})}$distal, then (X,S) is $\bigcup_{i=1}^{n} A_i^{(\overline{\overline{\mathcal{M}}})}$distal.
(Compare with Theorem 3).

Proof. In (1) and (2) we have (by (Sabbaghan and Shirazi, 2001a, Theorem 18) and Lemma 2):

$$\{e\} \subseteq \mathrm{J}\,(\mathrm{F}(\bigcup_{i=1}^{n} A_i\,, \mathrm{E}(X,S))) = \bigcup_{j=1}^{n} \mathrm{J}\,(\mathrm{F}(\bigcup_{i=1}^{n} A_i\,, \mathrm{E}(X,S_j)))$$

$$\subseteq \bigcup_{j=1}^{n} \mathrm{J}\,(\mathrm{F}(A_j\,, \mathrm{E}(X,S_j))) = \bigcup_{j=1}^{n} \{e\}.$$

So $\mathrm{J}(\mathrm{F}(\bigcup_{i=1}^{n} A_i\,, \mathrm{E}(X,S))) = \mathrm{J}(\overline{\mathrm{F}}(\bigcup_{i=1}^{n} A_i\,, \mathrm{E}(X,S_j))) = \{e\}$. Therefore (X, S) is $\bigcup_{i=1}^{n} A_i^{(\overline{\mathcal{M}})}$distal in (1) and (X,S) is $\bigcup_{i=1}^{n} A_i^{(\overline{\overline{\mathcal{M}}})}$distal in (2).

Theroem 8. Let A be a nonempty subset of X, we have:
1. $\mathrm{E}(X,S_0) \subseteq \mathrm{E}(X,S)$.
2. If (X,S) is distal, then (X,S_0) is distal.
3. If (X,S) is A-distal, then (X,S_0) is A-distal.
4. If (X,S) is $A^{(\overline{\mathcal{M}})}$distal and $A ? \overline{\mathcal{M}}\,(X,S_0)$, then (X,S_0) is $A^{(\overline{\mathcal{M}})}$distal.
5. If (X,S) is $A^{(\overline{\overline{\mathcal{M}}})}$distal and $A ? \overline{\overline{\mathcal{M}}}\,(X,S_0)$, then (X,S_0) is $A^{(\overline{\overline{\mathcal{M}}})}$distal.
6. Let Z be a closed invariant subset of (X,S), then Z is a closed invariant subset of (X, S_0) and $h_{(X,S)}(Z) \leq h_{(X,S_0)}(Z)$.
7. $\mathrm{P}(X, S_0) \subseteq \mathrm{P}(X, S)$, $\mathrm{P}_A(X, S_0) \subseteq \mathrm{P}_A(X, S)$ and $\overline{\mathrm{P}}_A(X, S_0) \subseteq \overline{\mathrm{P}}_A(X, S)$,

Proof. Take $S = S \cup S_0$ and use Lemma 2, Theorem 3 and Theorem 5.

Corollary 9. Let (X,S) be distal, then for each $s \in S - \{e\}$and each $m \in \mathbf{N}$, There exists a net $\{m_\gamma\}_{\gamma \in \Gamma}$ in $\mathbf{N}$ such that $\lim_{\gamma \in \Gamma} s^{m_\gamma} = s^{-m}$.

Proof. Let $m\in$ **N** and $s \in S - \{e\}$, by Theorem 8 $(X, \{s^K | K \in \mathbf{N}\}\cup\{e\})$ is distal, therefore $\mathrm{E}(X,\{s^k | k \in \mathbf{N}\}\cup\{e\})$ is a group, so there exists a net $\{m_{?\ddagger}\}_{\gamma\in\Gamma}$ in $\mathbf{N} \cup\{0\}$ such that $\lim_{?\S\Gamma} s^{m_{?\ddagger}} = s^{-m}$ $(s^0=e)$, since $s \neq e$ we can take $m_{?\ddagger}\in$ **N** $(\gamma\in\Gamma)$.

Corollary 10. Let ?>: $(X,S)\to (Y,S)$ be an onto homomorphism and let A be a nonempty subset of X and B be a nonempty subset of Y, then (with all the factors and extensions being under ?Y:

a. If (Y,S_0) is a proximal factor of (X,S_0), then (Y,S) is a proximal factor of (X,S).

b. If (Y,S_0) is an A-proximal factor of (X,S_0), then (Y,S) is an A-proximal factor of (X,S).

c. If (Y,S_0) is an $A^{(\overline{\mathrm{M}})}$proximal factor of (X,S_0), then (Y,S) is an $A^{(\overline{\mathrm{M}})}$proximal factor of (X,S).

d. If (X,S_0) is a proximal extension of (Y,S_0), then (X,S) is a proximal extension of (Y,S).

e. If (X,S_0) is a B-proximal extension of (Y,S_0), then (X,S) is a B-proximal extension of (Y,S).

f. If (X,S_0) is a $B^{(\overline{\mathrm{M}})}$ proximal extension of (Y,S_0), then (X,S) is a $B^{(\overline{\mathrm{M}})}$ proximal extension of (Y,S).

Proof. Use Theorem 8.

Theorem 11. Let A be a nonempty subset of X.

1. Let $a_1, \ldots, a_p, b_1, \ldots, b_q \in S$ be such that $S =(\bigcup_{i=1}^{p} S_0\, a_i) \cup (\bigcup_{i=1}^{q} b_i\, S_0)$, then:

a. $\mathrm{E}(X, S) = (\bigcup_{i=1}^{p} \mathrm{E}(X, S_0)\, a_i) \cup (\bigcup_{i=1}^{q} b_i \mathrm{E}(X, S_0))$.

b. (X,S) is distal if and only if $a_1 ,\ldots, a_p, b_1 ,\ldots, b_q$ are one to one and (X,S_0) is distal.

c. Suppose $a_1, \ldots, a_p, b_1, \ldots, b_q \in F(A,S)$. Then (X,S) is A-distal if and only if $a_1, \ldots, a_p, b_1, \ldots, b_q$ are one to one and (X,S_0) is A-distal.

d. Suppose $a_1, \ldots, a_p, b_1, \ldots, b_q \in F(A,S)$ and A?,$\overline{M}$ $(X,S) \cap \overline{M}$ (X,S_0). Then (X,S) is $A^{(\overline{M})}$distal if and only if $a_1, \ldots, a_p, b_1, \ldots, b_q$ are one to one and (X, S_0) is $A^{(\overline{M})}$distal.

e. Suppose $a_1, \ldots, a_p, b_1, \ldots, b_q \in \overline{\overline{F}}(A,S)$ and A?â$\overline{\overline{M}}$ $(X,S) \cap \overline{\overline{M}}$ (X,S_0) . Then (X, S) is $A^{(\overline{\overline{M}})}$distal if and only if $a_1, \ldots, a_p, b_1, \ldots, b_q$ are one to one and (X, S_0) is $A^{(\overline{\overline{M}})}$distal.

2. Let $s \in S$ be such that $s^{-1} \in S$, then:

a. $E(X, s^{-1}S_0s) = s^{-1}\, E(X, S_0)s$.

b. (X, S_0) is distal if and only if $(X, s^{-1}S_0s)$ is distal.

c. Suppose $s \in F(A,S)$. Then (X, S_0) is A-distal if and only if $(X, s^{-1}S_0s)$ is A-distal.

d. Suppose $s \in F(A,S)$ and A?$\$$$\overline{M}$ $(X,S_0) \cap \overline{M}$ $(X, s^{-1}S_0s)$. Then (X,S_0) is $A^{(\overline{M})}$distal if and only if $(X, s^{-1}S_0s)$ is $A^{(\overline{M})}$distal.

e. Suppose $s \in \overline{\overline{F}}(A,S)$ and A? $\overline{\overline{M}}$ $(X, S_0) \cap \overline{\overline{M}}$ $(X, s^{-1}S_0s)$. Then (X,S_0) is $A^{(\overline{\overline{M}})}$distal if and only if $(X, s^{-1}S_0s)$ is $A^{(\overline{\overline{M}})}$distal.

Proof.

1.

a. Let $r \in E(X,S)$, then there exists a net $\{s_{?‡}\}_{\gamma \in \Gamma} \subseteq S$ such that $\lim_{?ø\Gamma} s_\gamma = r$.

There exists a subnet $\{ s_{?‡_{?‡}} \}_{\lambda \in \Lambda}$ of $\{s_{?‡}\}_{\gamma \in \Gamma}$ and $\{ t_{?‡_{?‡}} \}_{\lambda \in \Lambda} \subseteq S_0$ such that:

$$(\exists i \in \{1,\ldots,p\} \;\; \forall \lambda \in \Lambda \;\; s_{?™_{?z}} ?‡_{?‡_{?‡}} a_i\,) \vee (\exists i \in \{1,\ldots,q\} \;\; \forall \lambda \in \Lambda \;\; s_{?‡_{?‡}}\, ?‡b_i\, t_{?‡_{?‡}}).$$

There exists a subnet $\{ t_{?‡_{?ð‡}} \}_{?‡\in\Omega}$ of $\{ t_{?‡_{?‡}} \}_{\lambda \in \Lambda}$ such that $\lim_{?ÐÐ‡} t_{?‡_{?ð‡}} \in$ $E(X,S_0)$, therefore $r \in (\bigcup_{i=1}^{p} E(X,S_0)a_i) \cup (\bigcup_{i=1}^{q} b_i\, E(X,S_0))$.

b. If (X,S) is distal, then $E(X,S)$ is a group, so $a_1, \ldots, a_p, b_1, \ldots, b_q$ are one to one, also (X,S_0) is distal by Theorem 8.

Conversely suppose $a_1, \ldots, a_p, b_1, \ldots, b_q$ be one to one and (X,S_0) be distal, then $E(X,S_0)$ is a group, so the elements of $(\bigcup_{i=1}^{p} E(X,S_0)a_i) \cup (\bigcup_{i=1}^{q} b_i E(X,S_0))$ are one to one, thus by (a) the elements of $E(X,S)$ are one to one and $J(E(X,S)) = \{e\}$. Therefore (X,S) is distal.

c. If (X,S) is A-distal and $a \in A$, then $F(a, E(X,S))$, is group and $a_1, \ldots, a_p, b_1, \ldots, b_q \in F(a, E(X, S))$, so $a_1, \ldots, a_p, b_1, \ldots, b_q$ are one to one, also (X,S_0) is A-distal by Theorem 8.

Conversely suppose $a_1, \ldots, a_p, b_1, \ldots, b_q$ be one to one and (X,S_0) is A-distal, then for each $a \in A$, then $F(a, E(X, S_0))$ is a group, so the elements of $(\bigcup_{i=1}^{p} F(a, E(X,S_0))a_i) \cup (\bigcup_{i=1}^{q} b_i F(a, E(X, S_0)))$ are one to one, thus the elements of $F(a, E(X, S))$, are one to one (by using (a) we have $(\bigcup_{i=1}^{p} F(a, E(X, S_0))a_i) \cup (\bigcup_{i=1}^{q} b_i F(a, E(X, S_0))) = F(a, E(X, S)))$ and, $J(F(a, E(X, S))) = \{e\}$. Therefore (X, S) is A-distal.

d. Use a similar method described in (c).

e. Use a similar method described in (c).

2. Use a similar method described in (1).

Corollary 12.

If S is a group and S_0 is a normal subgroup of S such that $\frac{S}{S_0}$ is finite, then (X,S) is distal if and only if (X,S_0) is distal.

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