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On the Solitary Waves in Arteries

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Abstract

Solitary waves are coincided with separatrices, which surround an equilibrium point with characteristics like a center, a sink, or a source. The existence of closed orbits in phase plane predicts the existence of such an equilibrium point. If there exists another saddle point near that equilibrium point, separatrix orbit appears. In order to prove the existence of solution for any kind of boundary value problem, we need to apply the fixed-point theorems. The Schauder's fixed-point theorem was used to show that there exists at least one nontrivial solution for equation of wave motion in arteries.

The equation of wave motion in arteries has a nonlinear character, and the amplitude of the wave depends on the wave velocity. There is no general analytical or straightforward method for prediction of the amplitude of solitary waves. Therefore, the solution must be found by numerical or non-straightforward methods. The methods of *saddle point trajectory*, *escape-time*, and *escape-energy* are introduced and shown that they are applicable methods with enough accuracy. Application of any of these approximate methods depends on the equation of motion, and the user preference.

Applying a phase plane analysis, it was shown that the domain of periodic solution is surrounded by a separatrix. The separatrix is coincident with the desired solitary wave. The amplitude of the solitary wave is the most important characteristic of the wave, and will be predicted with each of the above methods.

Keywords: *Solitary waves, Qualitative analysis, Fixed Point Theorems, Waves in arteries.*

1 Introduction

1.1 Solitary Wave

Solitary waves exhibit a particle-like behavior and decay to zero at infinity. They are solutions of nonlinear wave equations, however not all nonlinear wave equations have solitary wave solutions. Solitary waves are a class of nonlinear waves that have very interesting properties. They propagate without change of shape, and two solitons can cross without interaction. The properties of a solitary wave result from an exact balance between dispersion, which tends to spread the solitary wave into a train of waves, and nonlinear effects, which tend to shorten and steepen the wave.

For the Korteweg-deVries (KdV) equation ($u_t + uu_x + u_{xxx} = 0$), which is the first soliton to be noted in nature, the propagation speed of a solitary wave increases and the wave width decreases as the wave amplitude increases. Because large waves propagate faster than smaller waves, a large wave trailing a smaller wave will eventually catch up to the smaller wave. A complicated nonlinear interaction between the two waves results in a transfer of energy, mass and momentum from the larger wave to the smaller. Consequently, the rear wave shrinks in amplitude and slows down while the small one grows in amplitude and speeds up, propagating ahead of the trailing wave. The final large and small waves have exactly the same amplitudes as the initial large and small waves. The preservation of wave identities after a nonlinear interaction between two waves is a special property of the nonlinear wave equations.

The elevation u of a shallow solitary water wave traveling in the x direction, after ignoring Coriolis force and viscosity is given by, $u(x,t) = H \cdot Sech^2\left[n\left(x - \nu t - x_0\right)\right]$, where H is the maximum wave height, x_0 is the initial location of the solitary wave crest, and ν is the wave speed. Using η for the depth of water, the wave number n is defined by $n = \sqrt{3H^2 / 4\eta^3}$. While the length of a solitary wave is theoretically infinite, for practical purposes, the water surface elevation decays to zero fast with x and we can usually define a wavelength, L, as $L = 2\pi / n$. At a distance of $x = L/2$ away from the peak, the water surface displacement is reduced to 0.74% of its maximum value. Using the speed of a solitary wave, $\nu = \sqrt{g(\eta + H)} = 2\pi f$, the wave period can be defined as $T=L/C$. Wave frequency ω is related to wave number n by the dispersion relation, $\omega^2 = gn\tanh(n\eta) \approx gn^2\eta$, where g is the acceleration of gravity. Solitary waves are nonlinear because the amplitude of a given wave H is related to the wave speed ν and to the wave number n. Figure 1 depicts a solitary wave moving to the right.

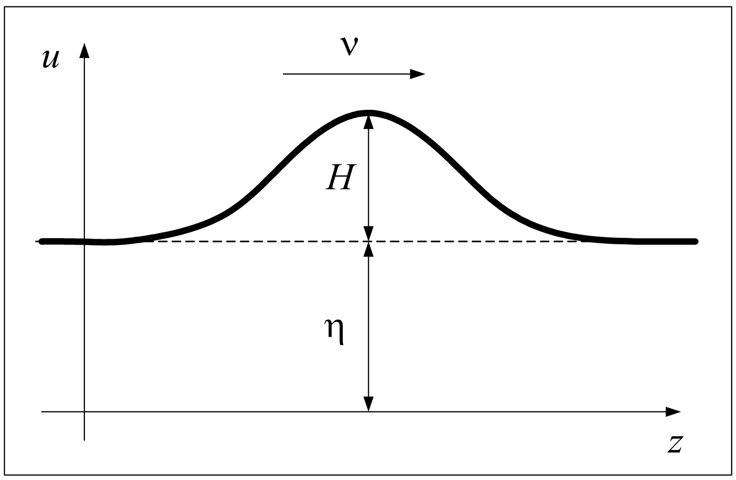

Figure 1- Schematic illustration of a solitary wave.

1.2 History

Solitons and scientific importance of solitary waves were discovered by Russell over one hundred and fifty years ago, when Russell, a young Scottish engineer, reported his scientific discovery to the British Association for the Advancement of Science that: “I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses on the Union Canal at Hermiston, when the boat suddenly stopped, the mass of water in the channel which it had put in motion accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.” By experiments, Russel discovered that the velocity ν of the wave depends on the maximum elevation H of the wave and on the depth η of the undisturbed water in the channel, (Russell 1844, Russell 1865). Independently, Boussinesq found the hyperbolic secant-squared solution (solitons) for the free surface waves, (Boussinesq). Later in 1895, Korteweg and deVries found the unidirectional equation of solitaries, which is now named after them, (Korteweg, and de Vries 1895). In less than 5 decades (1890-1940), soliton had been found in many other branches of science, (Newell 1983). Zabusky and Kruskal in 1965 demonstrated that solitary waves as solutions of the KdV equation can interact and carry on thereafter as if they never had interacted. Therefore, the word soliton was created to emphasize that it is a localized entity that may keep its identity after interaction, (Drazin 1983).

1.3 Application

Solitons are waves that move like particles along a conductor. An example is the ocean wave, which moves through the water-air interface without dissipating much of its energy and reverberates around the globe for an extended time without loosing its shape. This is in part due to the physical arrangement of the water molecules at the interface.

Communications engineers want light signals to travel through long fiber optic cables without changing shape, so they have been developing optical solitary waves that travel long distances without distortion. Solitary sound waves are thought to be very difficult to produce because the properties of air do not seem to permit them. Any large disturbance in air generates a traveling sound wave that changes shape as it propagates. Solitary waves in optical fibers maintain their shape because the material exhibits "dispersion," a speed of light that depends on frequency in just the right way but the speed of sound in air is relatively independent of frequency; so solitary sound waves have been a great challenge.

Over the last several years solitary waves generating in train tunnels created severe waveforms and problems for civil and traffic engineers. Several types of heavy machinery, such as compressors, generate acoustic shocks, which engineers consider a nuisance, and suppression of them is difficult.

The wide application of methods developed in the theory of oscillations and the wave theory were due to the progress in radio physics, plasma physics, and laser optics. In these applications, the most important problems were related to nonlinear waves in depressive media. Therefore, almost all the basic oscillation concepts, such as phase plane, self excitation, limit cycle, bifurcation, resonance, etc., have been widely used in the theory of waves. This was naturally accompanied by an intense development of approximate methods.

1.4 The Problem

The solution of the wave equation of the type $u(x, t)=u(z)$ where $z=x \pm nt$, indicates stationary traveling waves. Applying this substitution, the partial differential wave equations will transform into ordinary differential equations. The amplitude of linear waves as solutions of the linear PDE can be chosen arbitrary. On the contrary, the amplitudes of the nonlinear waves are determined by their nonlinear PDE. Consequently, period and wavelengths of periodic nonlinear waves depend on their amplitude. Typical forms of nonlinear waves are the solitary waves where the deterioration of the wave by dispersion is compensated by nonlinearity. Not every nonlinear PDE has solitary wave solution, and even if an equation does have one, it may not be possible to find the exact analytic form of the solitary wave nor its amplitude. Therefore, amplitude is the key factor in the analysis of solitary waves, and predication or evaluation of the amplitude of the solitary wave still is a challenge, (Kneubuhl 1997).

Solitary waves $u(x, t)$ can be characterized by the following conditions

$$u(x,t)=u(x \pm \nu t)=u(z) \tag{1}$$

$$\lim_{z \to -\infty} u(z)=c_1=cnst \quad , \quad \lim_{z \to +\infty} u(z)=c_2=cnst \tag{2}$$

$$u^2(x,t)=u^2(z) \le c_3^2 < \infty \tag{3}$$

where typical relations between c_1 and c_2 are $c_1=c_2=0$ as well as $c_1-c_2=\pm 2\pi$.

If the final equations are autonomous, one can use the method of phase trajectories, but the phase space of these equations are degenerated due to the wave velocity, ν. Therefore, all singularities, trajectories, separatrices, and limit cycles form a continuum. The separatrix are those trajectories which are going out of a saddle and returning to it (homoclinic) or entering another saddle (hetroclinic), (Lonngren, and Scott 1978). Solitary waves involve single-wave pulses with a bell-shaped profile propagating with constant speed. In general, determination of the properties of stationary solutions, including solitons, is a complicated problem. Predication of the wave height is an essential measure of the model accuracy, but waveform and phase are equally important, especially for wave interactions in 2-D problems. In a resent paper, Epstein and Johnston, presented a numerical scheme for predicting the amplitude of solitary waves in an elastic artery with any given speed of wave. They showed the importance and difficulties of finding the amplitude of a solitary wave in an elastic tube, (Epstein, and Johnston 1999).

The method of phase plane was introduced to the wave theory in the early sixties, and very soon was widely used for analyzing the behavior of shock waves, envelope waves and other types of solutions. Separatrices deserve special attention among phase trajectories on the wave phase plane. They illustrate the distinction in the roles of analogous types of solutions for the cases of oscillations and waves. A separatrix is a normalizable solution between the regions of phase space with topologically different types of trajectories, (Ostrovsky 1989).

2. Existence Theorem

Closed-form solution for most of the differential equations is limited. Because of that, in twentieth century, the approach of analysis of differential equations shifted to development of conditions that guarantee the existence of nontrivial solutions. In dynamics and vibration, periodic solutions are usually the most important solutions sought for a dynamic system. The significance of periodic solutions lies on the fact that all aperiodic responses, if convergent, would approach the periodic solutions at the steady state condition. Therefore, the periodic solutions would represent the steady-state response of the system.

Solitary waves are special periodic solution and the condition of their existence must be determined. In order to prove the existence of periodic solution in this study, an operator on a Banach space is defined to show that there exists a fixed point in the space under a defined operator. The defined operator would be based on the boundary value integral form of the problem.

Although the differential equation of the dynamic system to be considered is of the form $u''=f(u)$, we present and prove a stronger existence theorem and show that the required condition for a periodic solution exists. Consider the following general second order ordinary differential equation,

$u''=f(u,u')$

where f is a C^1 continuous real-valued function with domain R^2. It is smooth enough to ensure existence and uniqueness of the solution with any set of initial conditions. We establish the following Second Order Autonomous Existence theorem.

Second Order Autonomous Existence Theorem:
If there exist constants a and b, $a \geq b$, $k>0$, q and N, $q \geq N$, such that,

$$f(a,0) \leq 0 \leq f(b,0) \tag{4}$$

$$\frac{k(q-N)}{3} > M \tag{5}$$

where

$$N = max\{|a|,|b|\} \tag{6}$$

$$M = max\{|ku+f(u,u')| : (u,u') \in Z\} \tag{7}$$

$$Z = max\{(u,u') \in R^2 : |u-u_i| \leq 2q, |u'| \leq 2q\sqrt{k}\} \tag{8}$$

$$f(u_i,u'_i)=0,\ b \leq u_i \leq a \tag{9}$$

then there exists at least one $\tau \neq 0$ such that equation

$$u''=f(u,u') \tag{10}$$

has a nontrivial solution satisfying the following boundary condition,

$$u(0)=u(\tau) \tag{11}$$

The equilibrium point $u=u_i$ can be a center, sink, or source. If in addition, $f(0,0)=0$, and the point $u=0$ is a saddle point, then the system can have a separatrix, which originates at $u=0$, surrounds $u=u_i$ and terminates at $u=0$.

3 Solitary Wave in Artries

It was shown by Demiray (1996), that the equation of motion for wave propagation through a fluid-filled elastic tubes is,

$$u'' = f(u, u') = \frac{m}{s_z - mv^2}\left(\frac{s_\theta}{1+u} - \frac{1}{2}(p_0 + p)(1+u)\right) \tag{12}$$

where,

$$p = \frac{\nu^2}{2}\left(1 - \frac{1}{(1+u)^4}\right) \quad , \quad p_0 = 2s_\theta\big|_{u=0} \tag{13}$$

Equation (12) is an approximation in the sense that the amplitude and slope is assumed to be small everywhere, so u<<1, and terms proportional to $(\partial u / \partial z)^2$ could be neglected, (Epstein, and Johnston 1999). The explicit form of the functions s_θ, and s_z depend on the particular constitutive equation for the tube material. These are given by Demiray (1996), as:

$$s_\theta = \left[\lambda_\theta^2(1+u)^2 - \frac{1}{\lambda_\theta^2\lambda_z^2(1+u)^2}\right]e^{\alpha(I_1-3)} \tag{14}$$

$$s_z = \left[\lambda_z^2 - \frac{1}{\lambda_\theta^2\lambda_z^2(1+u)^2}\right]e^{\alpha(I_1-3)} \tag{15}$$

where

$$I_1 = \lambda_\theta^2(1+u)^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2\lambda_z^2(1+u)^2}\,. \tag{16}$$

The equation of motion (12) is in the form of the general equation (10) . As an example, we analyze the system for a typical human artery to show the application of the theorem. The following numerical values are provided in the order of magnitude of actual biological measurements of human arteries, (Yomoza 1987).

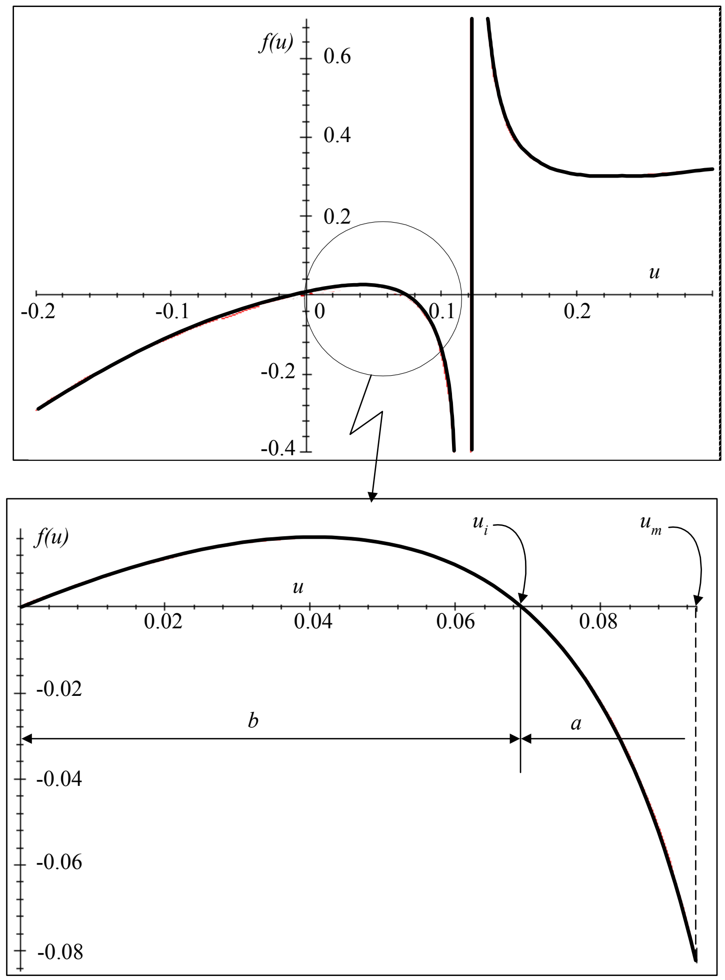

Figure 2- Plot of the right hand side of equation (12) indicating that the condition (4) is fulfilled

$$\alpha = 1.948 \qquad m = 0.4 \qquad \lambda_\theta = 1.2 \qquad \lambda_z = 1.5 \qquad \nu = 8 \tag{17}$$

Figure 2 depicts the right hand side of Equation (12). It indicates that the condition (4) is fulfilled, and illustrate the domain of *a* and *b*. Choosing *k=1*, determines that,

$$N = max\{|a|,\ |b|\} = a < u_m \tag{18}$$

where $u_m \approx 0.063869$ is the amplitude of the solitary wave. The graph of *u+f* is shown in Figure 3 and is used to find *M* as:

$$M = max\{|u + f(u,u')| \ :\ (u,u') \in Z\} \approx 0.070268\,. \tag{19}$$

Thus, if *q* is any number satisfying the following inequality

$q > 3M + a$

then, there exists at least one initial condition in the following rectangle in state space, which satisfies the condition (11) and generates a periodic orbit.

$$|u-u_i| \le 2q \quad , \quad |u'| \le 2q$$

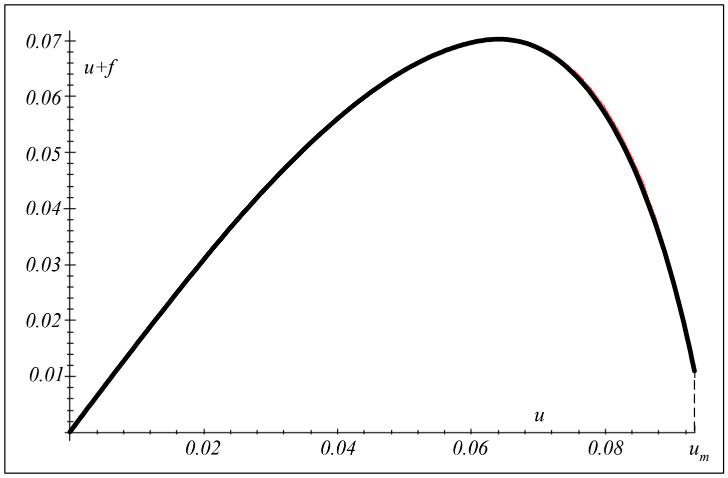

Figure 3- Illustration of u+f

4. Approximation of Amplitude

4.1 Phase Plane Method

The method of phase plane was introduced to the wave theory in the early sixties, and very soon was widely used for analyzing the behavior of shock waves solitons, envelope waves and other types of solutions. Separatrices deserve special attention among phase trajectories on the phase plane of the wave. They illustrate the destination in the role of analogous types of solutions for the cases of oscillations and waves. A separatrix is a normalizable solution between the regions of phase space with topologically different types of trajectories, (Novikov, Novikov, and Manakov 1984). In this section, we analyze the system in phase plane and look for some numerical method to evaluate the amplitude u_m of the solitary wave in arteries.

The phase portrait and tangent field of equation (12) is shown in Figure 4. The two orbits that are plotted in solid line, pass through points (0.091,0) and (0.095,0). Their time histories are also illustrated in Figure 5.

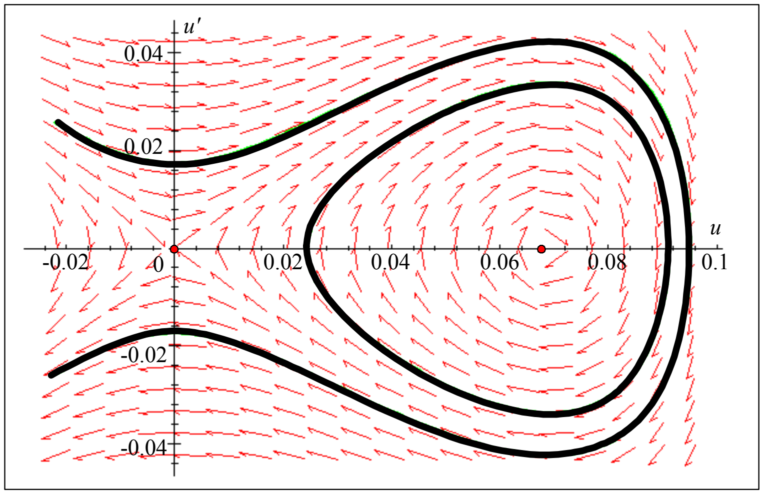

Figure 4- Phase portrait and tangent field of equation (16)

It is seen from Figure 4 that the origin is a saddle point and point (u_i, 0) is a center. The characteristic of equilibrium points would also be predictable by considering Figure 2. The saddle point trajectory separates two different motions, and indicates the solitary wave. Here the saddle point orbit is a homoclinic one since it leaves the equilibrium and returns asymptotically to it as time increases.
We would like to find the amplitude, u_m, of the solitary wave, which is the point of intersection of the saddle point orbit with u-axis.

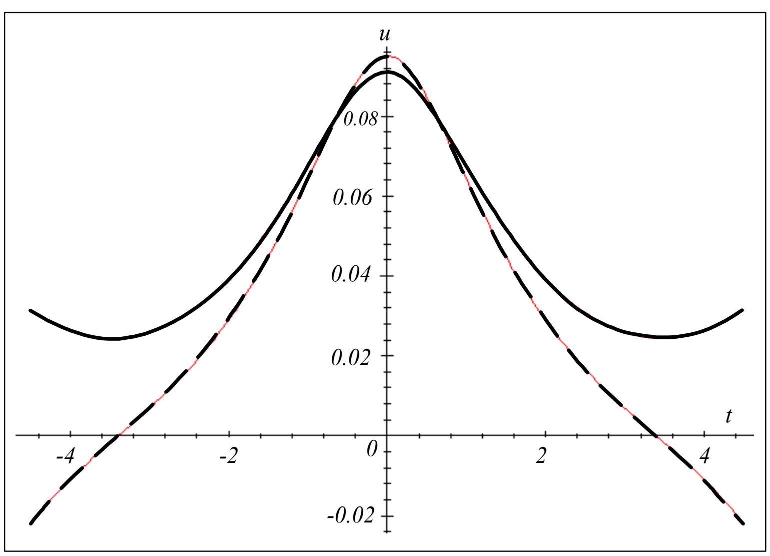

Fig. 5- Time history of the orbits shown in Fig. 3

It is known that if the equation of motion is in the form of $u''=f(u)$, then the integral of $f(u)$ between 0 and u_m must vanish, (Epstein, and Johnston 1999). Thus u_m, which is shown in Figure 2, could be found by satisfying the following equation:

$$\int_0^{u_m} f(u)du = 0 \tag{20}$$

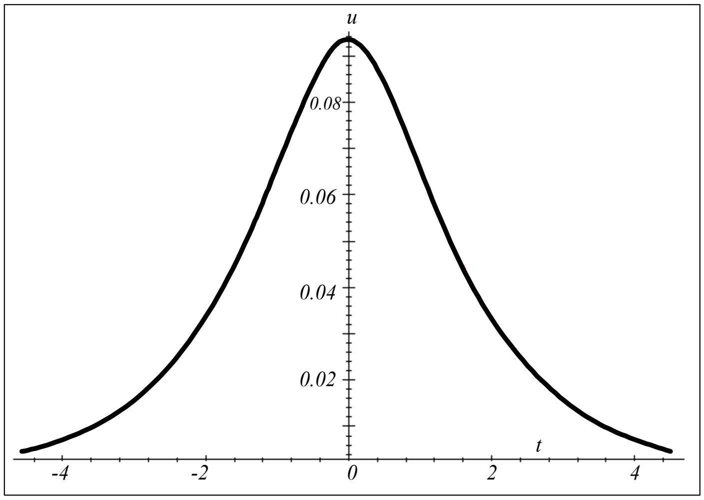

Figure 6- solitary wave for equation (16)

The solitary wave must sit between orbits of Figure 4 and/or graphs of Figure 5. The time history of the solitary wave is illustrated in Figure 6.

In order to find the amplitude of the solitary wave, u_m, the saddle point trajectory is searched for. To do this, the linearized state equations around the saddle point, *(0, 0)* are analyzed.

$$\begin{aligned} \dot{u} &= y \\ \dot{y} &= \left.\frac{\partial f}{\partial u}\right|_{(0,0)} u + \left.\frac{\partial f}{\partial y}\right|_{(0,0)} y = 0.68147u \end{aligned} \tag{21}$$

Hence, the eigenvalues and eigenvectors of the system are

$$\pm 0.8255121 \qquad \begin{Bmatrix} 0.7711793774 & -.785400456\ 6 \\ 0.6366179135 & 0.6483575866 \end{Bmatrix} \tag{22}$$

Due to physical appearance of the waves in arteries, we are only concerned with the positive half space $u>0$,. One of the eigenvectors shows the direction of departure from saddle point, and the other one shows the direction of arrival to the saddle point. We call the first one, positive eigenvector and the latter, negative eigenvector.

If we disturb the states of the system from unstable saddle point equilibrium on the positive eigenvector direction and set the system to be released from the following initial conditions,

$$\begin{aligned} u(0) &= 0.00001 \times 0.7711793774 \\ y(0) &= 0.00001 \times 0.6366179135 \end{aligned} \tag{23}$$

then, the states of the system will change due to the nonlinear equation of motion, and will trace the saddle point trajectory. The intersection of the trajectory with u axes will determine the amplitude of the solitary wave u_m. Figure 7 shows the saddle point orbit, and its time history is illustrated in Figure 8. The amplitude of the wave is approximately equal to u_m=*0.063869*.

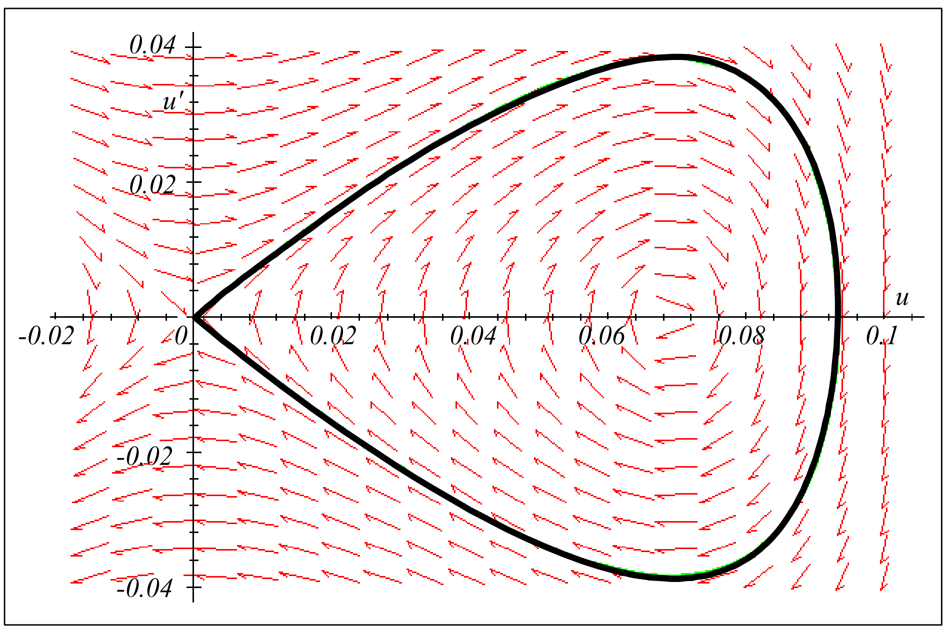

Fig. 7- phase plane illustration of saddle point orbit

Disturbance of a system from a saddle point equilibrium in direction of positive eigenvector which is used to find the approximate amplitude of the solitary wave, is not only applicable to equation (12), but also to any equation of the form $u'' = f(u,u')$.

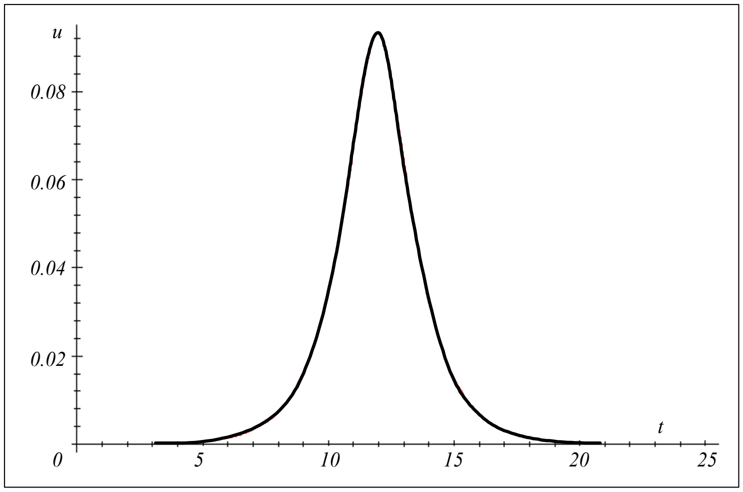

Figure 8- Time history of the response of the system, disturbed from saddle point equilibrium

4.2 Escape Energy Method

An alternative method to evaluate u_m, is called Escape-Energy method. If we write the equation of motion (12) in the following form:

$$u''+u = u + f(u) \tag{24}$$

then, it can be seen that

$$\frac{1}{2}E' = \frac{1}{2}\frac{d}{dz}\left(u'^2+u^2\right) = [u + f(u)]u' \tag{25}$$

therefore,

$$E = 2\int [u + f(u)]u'dz \tag{26}$$

where E is called the moving energy. A disturbed system on a positive eigenvector ripples from the saddle point easier than closed by disturbed points. Therefore, the value of moving energy is minimum for a saddle point orbit. Figure 9 illustrates the required energy for the system that starts from ($u(0) = u_0 > u_i$, $u'(0) = 0$) and escapes from second quarter of phase space, *($u>0$, $u'<0$)*. Figure 9 indicates that the escape energy is minimum for the saddle point trajectory. Hence, upon determining the position of saddle equilibrium, the value of moving energy can be determined for disturbed system. The minimum of the moving energy corresponds with the solitary wave.

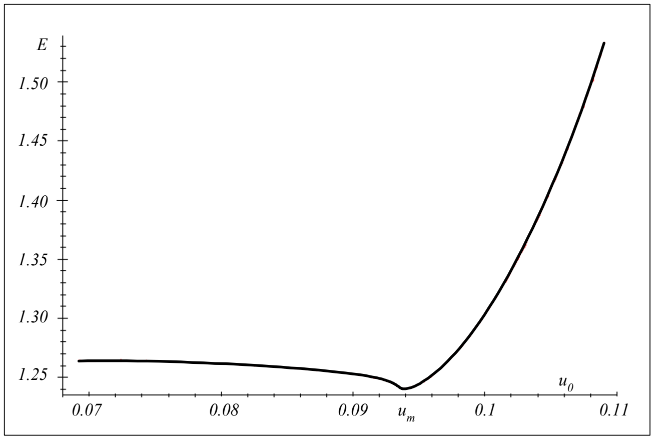

Figure 9- Escape-energy E versus u0

4.3 Escape Time Method

Due to the dynamic behavior of saddle point trajectory, the required time to close the solitary orbit is infinity theoretically. The third approximate method, which is presented here, is called the Escape Time method. Based on Equation (12), the time of motion from the following initial conditions

$$u(0) = u_0 > u_i \quad , \quad u'(0) = 0 \tag{27}$$

where u_i is the position of the center equilibrium point, is

$$T = \int \frac{du}{\sqrt{2\int f(u)du}} \,. \tag{28}$$

The time required to escape from second quarter of phase space, *(u>0, u'<0)*, will be maximum for the saddle point trajectory, since the saddle point trajectory approaches the equilibrium point asymptotically. It is shown in Figure 10, that the escape time *T*,

$$T = \int_{\substack{u=u_m \\ u'=0}}^{\substack{u=0 \\ u'=0}} \frac{du}{\sqrt{2\int f(u)du}} \tag{29}$$

is maximum for u_i .

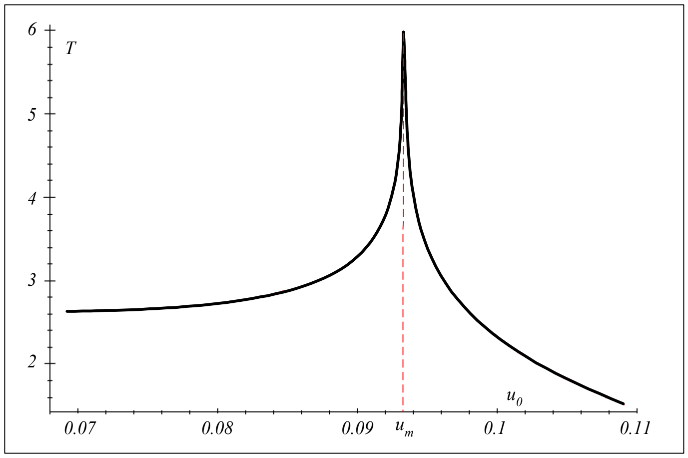

Figure 10- Escape-time T versus u0

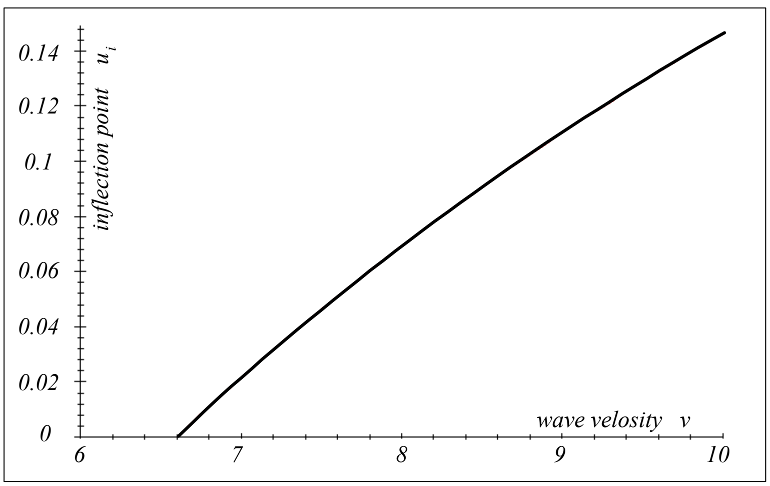

Figure 11- Position of the inflection point ui

4.4 Effect of Wave Speed

The approximate methods presented in previous sections are applicable to any system for which the origin is a saddle point equilibrium. Hence, upon determining the position of saddle point, it must be transformed on the origin. The behavior of the equation of motion (12) depends on the wave speed v. In order to analyze the effect of the wave speed, the position of the equilibrium point u_i, and the amplitude of solitary waves u_m, are determined for different wave speed and are illustrated in Figure 11 and 12. A critical wave speed is the minimum required wave speed to have a solitary wave.

Employing Equations (12) to (16) along with condition (20) provides and equation for the critical velocity to have a solitary wave.

$$v_c = \frac{\sqrt{2\left(\alpha\lambda_\theta^8\lambda_z^4 + 2\alpha\lambda_\theta^4\lambda_z^2 + 2\lambda_\theta^2\lambda_z^2 + \alpha\right)}}{\lambda_z^2\lambda_\theta^2} e^{\left(\frac{\alpha\left(\lambda_\theta^4\lambda_z^2 + \lambda_\theta^2\lambda_z^4 - 3\lambda_\theta^2\lambda_z^2 + 1\right)}{2\lambda_\theta^2\lambda_z^2}\right)} \tag{30}$$

Dependency of the amplitude of solitary wave to the wave speed, and the critical wave speed, v_c=*6.6,* are shown in Figure 12.

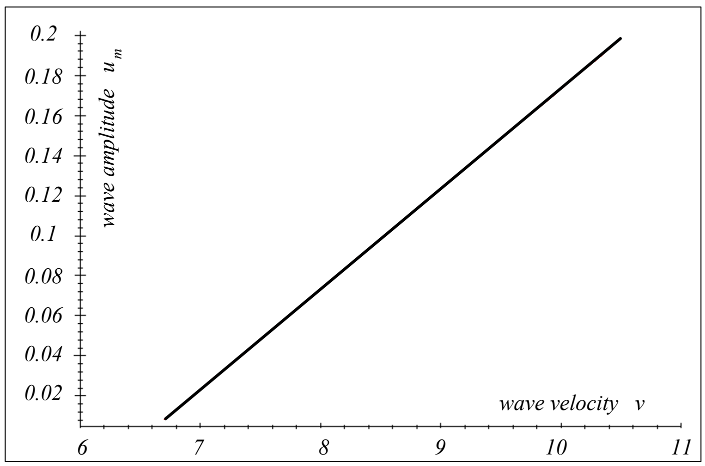

Fig. 12- dependence of the amplitude um to v

5. Model Analysis

The equation of motion (12) is completely defined whenever the explicit form of the stress components s_z and s_θ are derived. For a symmetrically deformed membrane, the components of the principal stresses may be given as:

$$\sigma_z = \Lambda_z \frac{\partial \Sigma}{\partial \Lambda_z} \quad , \qquad \sigma_\theta = \Lambda_\theta \frac{\partial \Sigma}{\partial \Lambda_\theta} \tag{31}$$

where

$$\sigma_z = \mu\, s_z \quad , \qquad \sigma_\theta = \mu\, s_\theta \tag{32}$$

and

$$\Sigma = \Sigma(\Lambda_z, \Lambda_\theta) = \Sigma(\lambda_z, u) \tag{33}$$

Σ is the strain energy density function of the tube material. It is seen that s_z and s_θ are functions of *u(z, t)*, but the form of their functions depend on the mathematical model of strain energy density function, Σ. Equations (31) and (32) are derived from Demiray's model D1,

$$\Sigma_{D1} = \frac{\mu}{2\alpha}\left(e^{\alpha(I_1-3)} - 1\right) \tag{34}$$

Solitary wave has one bump and then exponentially tends to zero. Therefore, outside a small interval where the pulse exists, the function goes to zero. The phase plane analysis showed that the function *f(u)* must have a double zero at $u=0$ and another at $u=u_i>0$, and must be C^1 in this interval. It must be positive in the interval $u \in (0, u_i)$ and negative in the interval $u \in (u_i, u_m)$. The separatrix orbit, which originates at $u=0$, surrounds the other equilibrium point u_i, and separates the domain of closed orbits around u_i and open orbits. In addition, the integral of *f(u)* between $u=0$ and $u=u_m$ must vanish.
If we rewrite the equation of motion (12), in the following form

$$u'' = \frac{m}{\Lambda_z \dfrac{\partial\Sigma}{\partial\Lambda_z} - \mu m v^2}\left(\frac{\Lambda_\theta \dfrac{\partial\Sigma}{\partial\Lambda_\theta}}{1+u} - \frac{1}{2}(p_0 + p)(1+u)\right) = f(u) \tag{35}$$

and enforce the conditions $f(0)=0$ and $f(u_i)=0$, the following conditions on the function □ are achieved:

$$\left[\Lambda_\theta \frac{\partial\Sigma}{\partial\Lambda_\theta}\right]_{u=0} = \frac{\mu}{2} p_0 \tag{36}$$

$$\left[\Lambda_\theta \frac{\partial\Sigma}{\partial\Lambda_\theta}\right]_{u=u_i} - (1+u_i)^2\left[\Lambda_\theta \frac{\partial\Sigma}{\partial\Lambda_\theta}\right]_{u=0} = \frac{\mu}{2} p_i (1+u_i)^2 \tag{37}$$

where

$$p_i = \frac{v^2}{2}\left(1 - \frac{1}{(1+u_i)^4}\right) \tag{38}$$

The continuity condition shows that

$$\left[\Lambda_z \frac{\partial\Sigma}{\partial\Lambda_z}\right] \neq mv^2 \quad , \quad 0 < u < u_m \tag{39}$$

and in addition,

$$\left[\frac{\partial f}{\partial u}\right]_{u=0} > 0 \quad , \quad \left[\frac{\partial f}{\partial u}\right]_{u=u_i} < 0 \tag{40}$$

and the integral condition leads to the following equation:

$$\int_{u=0}^{u=u_m} f(u,u')du = 0 \tag{41}$$

Now we examine the following three energy functions, presented by Demiray (1972),

$$\Sigma_{D1} = \frac{\mu}{2\alpha}\left(e^{\alpha(I_1-3)} - 1\right) \tag{42}$$

Ishiara (1951),

$$\Sigma_I = \frac{\mu}{2}\left[\beta(I_1-3) + (1-\beta)(I_2-3) + \delta(I_1-3)^2\right] \tag{43}$$

and Demiray (1976),

$$\Sigma_{D2} = \frac{\mu}{2\alpha}\left(e^{v(I_1-3)} - 1\right). \tag{44}$$

The functions Σ_{D1}, Σ_{D2}, satisfy all the required conditions, and Σ_I could only be a satisfactory energy function for some β and δ.
Now, suppose that the function $f(u)$ be a given function, say

$$u'' = f(u) = -A(v)u^2 + B(v)u \tag{45}$$

$$\forall v:\ A(v) > 0,\ B(v) > 0 \tag{46}$$

This function satisfies the conditions (40) and (41), provided that

$$u_m = \frac{3B(v)}{A(v)}. \tag{47}$$

Substitution of (45) into (35), reduces Equation (35) to the following parametric partial differential equation which can be used to determine the energy function Σ.

$$K(\Lambda_z,\Lambda_\theta)\frac{\partial\Sigma}{\partial\Lambda_z} - G(\Lambda_z,\Lambda_\theta)\frac{\partial\Sigma}{\partial\Lambda_\theta} = H(\Lambda_z,\Lambda_\theta) \tag{48}$$

There exists no general method for solving this differential equation. Thus to find a satisfactory energy function one must consider physical features in addition to the mathematical considerations.

Conclusion

The type of solitary waves in arteries and tubes depend on the given model for strain energy density function of the tube material. There are some necessary physical conditions, which must be satisfied by the strain energy function. Once a strain energy function is defined, the wave equation of motion will be set up. The principal characteristic of solitary waves is their amplitude. Although there is no general analytical method, the amplitude can be found by some numerical and non-straightforward methods. Three approximation methods are explained and examined in this paper to evaluate the amplitude of solitary waves.

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Nomenclature

a, b, c	constant parameters
C	wave velocity
C^1	class of continues first derivative function
E	escape moving energy
f	wave function, frequency [Hz]
h	tube wall thickness
g	acceleration of gravity
H	maximum wave height
I_1, I_2	basic invariant of the Green deformation tensor
k	restoring coefficient constant
L	wavelength
n	wave number
p	total inner pressure
R	initial radius of tube, set of real numbers
s	dummy variable
s_z, s_θ	stress components
S, B	Banach spaces
t	Time
T	wave period
u	radial displacement
u_i	Inflection point
U	operator
v	wave velocity
z	axial coordinate
α, m	material constants
Γ	escape moving time
λ_z	axial stretch ratio
λ_θ	circumferential stretch ratio
Λ_z	stretch in axial direction
Λ_θ	stretch in circumferential direction
ρ	mass density of tube material
ω	wave frequency [rad/s]
τ	period
ρ_f	mass density of fluid
σ_z, σ_θ	total Cauchy stresses
μ	shear modulus of tube material
η	depth of the water
Σ	strain energy
subscripts	
m	maximum
z	axial
θ	circumferential
1, 2	invariant indication