*Iranian Int. J. Sci.* ***6****(1), 2005, p.73-84*

# Two Exterior Griffith Cracks Opened By Heated Wedge in an Infinite and Isotropic Medium

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*(received: 15/11/2003 ; accepted: 5/7/2004)*

## Abstract

In this paper we have investigated the behavior of two exterior Griffith cracks opened by heated wedge in an infinite and isotropic medium.

By using Fourier sine and cosine transformation under plane strain condition we have obtained the closed form expressions for the stress intensity factor and the crack shape for the problem. Two special cases of heat distributions are discussed, when wedge geometry is prescribed.

***Keywords:*** *Griffith crack, Stress intensity factors, Fourier transformation, Heated wedge, Normal stress components.*

## 1. Introduction

A very rapid development of thermo elasticity has been considered after the Second World War by various engineering sciences. Griffith who was known as father of fracture mechanics considered the mathematical theory of elasticity of crack problems in his two pioneering papers published in 1920 and 1924.

A problem of an interior Griffith crack opened by a heated wedge in an infinite strip whose edges are parallel to crack axis has been recently published by Saraj (Saraj, 1994). Kushwaha has been introduced a new approach in investigating the problem of stress field in the neighborhood of Griffith crack (Kushwaha, 1974). The problem of stress intensity factors for a Griffith crack opened by thermal stresses in an infinite strip is discussed by Kushwaha and Umesh (Kushwaha and Chandra, 1982). Problem of crack opening due to stresses on crack faces as well as the neighborhood of the crack, by Lowengrub (Kushwaha and Chandra, 1982; Sneddon, 1946).

A note on Griffith cracks is investigated by Lowengrub (Lowengrub, 1966). An excellent survey of the crack problems in the theory of elasticity can be seen by Sneddon and Lowengrub (Sneddon and Lowengrub, 1970). Stress intensity factors for an interior Griffith crack opened by heated wedge in a strip whose edges are normal to crack axis is recently published by Saraj (Saraj, 2001).

The title problem can also be assumed as an infinite number of exterior Griffith cracks which are equally spaced from each other in the medium, see FIGI. And we reduce the above problem to the problem of two exterior cracks in an infinite medium, see FIGII. While the Griffith crack occupies the spaces $b \leq |x| \leq c$ ($y$=0) and symmetrical with respect to y-axis with the following boundary conditions

$$\sigma xy\,(x, 0) = 0 \quad 0 \leq |x| < \infty \tag{1.1}$$

$$\sigma yy\,(x, 0) = 0 \quad b < |x| < c \tag{1.2}$$

$$u_y\,(x, 0) = \begin{cases} u_0\,(x) & c \leq |x| < \infty \\ 0 & 0 \leq |x| < b \end{cases} \tag{1.3}$$

Where ($\sigma$xx, $\sigma$xy, $\sigma$yy) and ($u_x$, $u_y$) are components of stresses and of the displacement vector respectively.

$U_0$ ($x$) is a wedge shape function, and we assume that the thermal and elastic properties of the medium do not change with heat variation, and also all the physical quantities vanishes as $|x| \to \infty$.

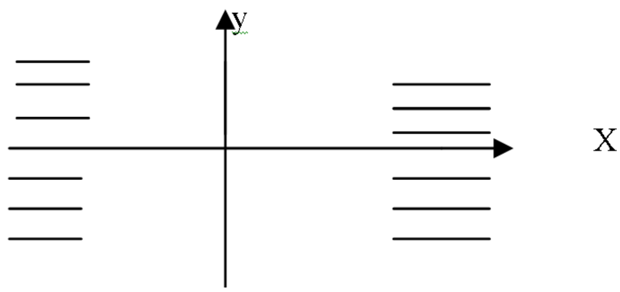


**Figure 1- Infinite number of exterior Griffith cracks equally spaced along Y-axis.**

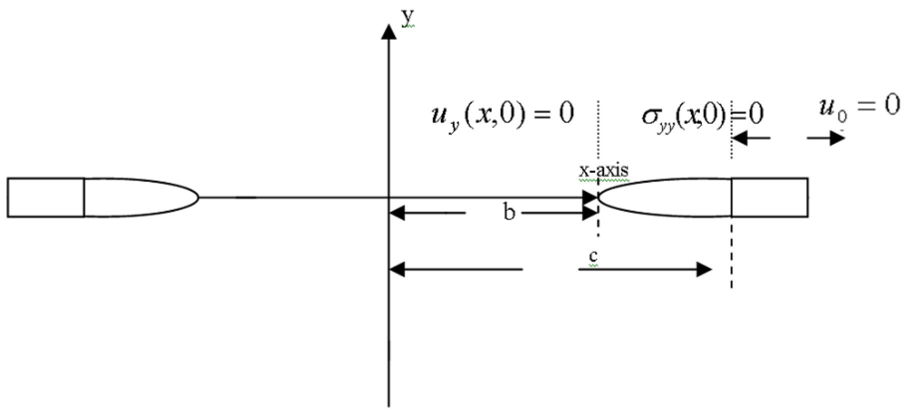


**Figure 2- Crack opening due to heat with boundary conditions for elasticity problem.**

Since the problem is linear we assume that stresses developed by temperature variation opens out the Griffith cracks as it is given through the boundary conditions (1.1) - (1.3). No heat sources or sinks are assumed in the medium and the medium is assumed under plane strain condition.

Throughout the analysis it has been checked that the crack faces do not meet other than the crack tips. (See Burniston, 1969), $u_y$ $(x, 0)>0$ $b$ ).

$$b<|x|<c... \tag{1.4}$$

we use the following definition for infinite Fourier sine and cosine transform

$$f_{cs}(\xi , \zeta )= \int_0^\infty \int_0^\infty f(x, y) \cos (\xi x) \text{Sin} (\zeta y)\, dx\, dy \tag{1.5}$$

## 2. Formulation

The physical problem is reduced to the solution of the following pair of simultaneous partial differential equation in absence of body forces,

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0 \tag{2.1}$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}=0 \tag{2.2}$$

with stress-strain relations as

$$\sigma_{ij}=2\mu e_{ij}+\lambda(e_{kk}-\gamma T)\,\delta_{ij}, \qquad i,j=x,\,y \tag{2.3}$$

$$e_{ij}=\frac{1}{2}\,(u_{i,j}+u_{j,i}),\ \gamma=(3\lambda+2\mu)\alpha \qquad I,\,j=x,\ y \tag{2.4}$$

where $\gamma$ and $\mu$ are Lame's constant, $\alpha_t$ is the coefficient of linear expansion, $\delta_{ij}$ is the Kronecker delta and T satisfies the Laplace's equation

$$\nabla^2 T=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)T=0, \tag{2.5}$$

for simplicity $\mu$ is taken as the unit of stress. Substituting for the stress components in terms of the displacement from (2.3) and (2.4) in (2.1)-(2.2) and solve for $u_x$ we get

$$\nabla^4 u_x=0. \tag{2.6}$$

The present problem is solved in two stages.
**Stage -A** (Temperature distribution problem).
**Stage -B** (Elasticity problem).

In **stage-A** of the problem we see that the temperature which is distributed over the surface of the crack $b\leq |x| \leq c$ due to the heated wedge, causes the further opening of the crack, whose solution is obtained by solving (2.5). In **stage-B** of the problem we deal with the problem of elasticity in which we apply the method of Kushwaha (Kushwaha, 1974) to solve the fourth order homogenous partial differential equation in (2.6), which represent the solution of elasticity problem, while the solution of Laplace's equation in (2.5) represents the solution of temperature distribution with the following conditions.

$$T(x,0)=\begin{cases}\theta(x) & c\le |x|<\infty\\ 0 & 0\le |x|\le b\end{cases} \tag{2.7}$$

$$\frac{\partial T(x,y)}{\partial y}\Big|_{y=0}=f(x) \qquad b<|x|<c \tag{2.8}$$

where $\theta(x)$ and $f(x)$ are known functions. The other component of displacement $u_y$ is obtained in terms of $u_x$ through the equations (2.1)-(2.2) as

$$u_y(x,y)=\frac{1}{1-\alpha^2}\left[\alpha^2\int\frac{\partial u_x}{\partial x}dy+\int\frac{\partial u_x}{\partial y}dx\right] \tag{2.9}$$

with $$\alpha^2=\frac{2(1-\sigma)}{1-2\sigma}=\frac{\lambda+2\mu}{\mu}$$

and $\sigma$ as the Poisson's ratio of the medium.

**3. Reduction to Triple Integral Equations**

**Stage-A:** We take the Fourier cosine transform of the equation (2.5) w.r.t x and then inverting we get .

$$T(x,y)=\int_0^\infty A(\xi)e^{-\xi y}\cos(\xi x)\,d\xi \tag{3.1}$$

where A is a constant to be determined. Since the geometry of the problem is symmetrical, therefore the boundary conditions (2.7) and (2.8) on using (3.1) are respectively reduced to

$$\int_0^\infty A(\xi)\cos(\xi x)\,d\xi=\begin{cases}\theta(x) & c\le |x|<\infty\\ 0 & 0\le |x|\le b\end{cases} \tag{3.2}$$

$$\int_0^\infty \xi A(\xi)\cos(\xi x)\, d\xi = -f(x) \qquad b<x<c \tag{3.3}$$

Stage-B: We follow the method of kushwaha (Kushwaha, 1974). We take Fourier sine transform of the equation in (2.6), on solving and inverting, we get.

$$u_x(x, y) = \int_0^\infty [B(\xi)+yE(\xi)]e^{-\xi y}\,\mathrm{Sin}\,(\xi x)d\xi \tag{3.4}$$

where B and E are constants to be determined. On solving the boundary condition in (1.1), we obtain

$$E(\xi)=\xi(1-B^2)B(\xi) \tag{3.5}$$

and the boundary conditions (1.2) - (1.3) is reduced to the following triple integral equations,

$$\int_0^\infty -B(\xi)\cos(\xi x)d\xi = {}^{-2}\begin{cases} u_0(\mathrm{x}) & \mathrm{c} \le |\mathrm{x}| \le \infty \\ 0 & 0 \le |x| \le b \end{cases} \tag{3.6}$$

$$\int_0^\infty \xi B(\xi)\cos(\xi x)d\xi = \frac{(3?‡^2-4)?X}{2(?G^2-1)}\, T(x, 0)\,, \qquad b<x<c \tag{3.7}$$

**4. Solution of Triple Integral Equations**

The trial solution for triple integral equations (3.2) - (3.3) and (3.6) - (3.7) is sought with the help of method of Srivastava and Lowengrub (Srivastava, 1970), we take the trial solution for (3.2) as

$$A(?\grave{a}) = \frac{2}{?J}\left[\int_b^c h_0(t)\sin(?\grave{e})dt - \int_c^\infty ?\hat{a}(t)\sin(?\text{Ð})dt\right] \tag{4.1}$$

on using the property

$$\int_0^{\infty} \frac{\sin(?@) \cos(?@) d?@}{?\text{¸}} = \begin{cases} \frac{?r}{2} & t > x \\ 0 & t > x \end{cases} \tag{4.2}$$

(3.2) is satisfied identically if

$$\int_b^c h_0(t)dt = -\theta(c) \tag{4.3}$$

where θ(c) is the temperature at the tip “c” of the crack.
Substituting for A(ξ) from (4.1) into (3, 3) and applying the formula

$$\int_0^{\infty} \frac{\sin\ (?.\ t)\sin(?.x)}{?\S} d?è = \frac{1}{2}\log\left|\frac{t+x}{t-x}\right| \tag{4.4}$$

we get

$$h_0(t) = \frac{1}{?‡^2\ ?5(t)} \left[\int_b^c \frac{x?¼(x)\ p(x)}{t^2 - x^2}\ d\ x + D\right] \tag{4.5}$$

where

$$\delta(t) = \left[\left|(t^2 - b^2)(c^2 - t^2)\right|\right]^{\frac{1}{2}} \tag{4.6}$$

and

$$p\ (x) = \frac{2}{?Ó}\int_c^{\infty} ?‡(t)\frac{tdt}{t^2 - x^2} - f\,(x) \tag{4.7}$$

where the constant D can be determined through the conditions (4.3)-(4.5).
Similarly the trial solution for (3.6) is taken as

$$-B\ (\xi) = \frac{2}{?9\!\!\!1}\left[\int_b^c h_1(t)\sin(?ï)dt - ?\text{Ï}^{-2}\int_c^{\infty} u_0'\ (t)\sin(?¢)dt\right] \tag{4.8}$$

then (3.6) is satisfied identically if

$$\int_b^c h_1\ (t)\ dt = {}^{-2}u_0(b) \tag{4.9}$$

and similarly the substitution of (4.8) in (3.7) yields the result

$$h_1(t) = \frac{1}{?K^2?K(t)}\left[\int_b^c \frac{x?X(x)\,p_1\,(x)dx}{t^2 - x^2} + D_1\right] \tag{4.10}$$

where $D_1$ is an arbitrary constant and

$$p_1(x) = \frac{(3?^{2} - 4)?^{\cdot\cdot}_{t}}{2(?\$ - 1)} T(x,0) + \frac{2}{?\$} ?m^{-2} \int_c^{\infty} \frac{t\, u'(t)dt}{t^2 - x^2} \qquad (4.11)$$

**5. Physical Quantities**

To calculate the prescribed temperature in $b < x < c$ (y=0) we evaluate the equation in (3.2) and we get

$$T(x, 0) = \int_x^c h_0(t)\, dt \qquad b < x < c \qquad (5.1)$$

where $h_0(t)$ is given by (4.5)

**Crack Shape**

The crack opening displacement $u_y(x, 0)$ is obtained by evaluating the integral in (3.6) for the interval $b < x < c$ and it is obtained as

$$u_y(x, 0) = 2\int_x^c h_1(t)dt \qquad b < x < c \qquad (5.2)$$

where $h_1(t)$ is given by (4.10).

**Normal Stress Components**

$\sigma yy(x, 0)$ is obtained by evaluating the integral in (3.7) for the intervals $0 \le x \le b$ and $c \le x < \infty$ . The value of temperature is known for the above intervals, so we get

$$\sigma yy(x, 0) = \frac{2}{?3}\left[ \pm \frac{1}{?3^{2}\, ?3_1(x)} \int_0^b \frac{x?\!\!<\!(x)\, P_1(x)}{t^2 - x^2} - ?\mathring{A}^{2} \int_c^{\infty} \frac{t u_0'(t)dt}{t^2 - x^2} \right] \qquad (5.3)$$

($\pm$) signs are taken for $0 \le x \le b$ and $c \le x < \infty$ respectively, where $\delta_1(x)$ is given by

$$\delta_1(x) = \left[(x^2 - b^2)(x^2 - c^2)\right]^{\frac{1}{2}} \qquad |x| \ge c$$

$$\delta_1(x) = \left[(b^2 - x^2)(c^2 - x^2)\right]^{\frac{1}{2}} \qquad |x| \le b$$

**Stress Intensity Factors**

They are defined by

$$K_b = \lim_{x \to b^-} \sqrt{b - x}\ \sigma_{yy}(x, 0) \tag{5.4}$$

$$K_c = \lim_{x \to c^+} \sqrt{x - c}\ \sigma_{yy}(x, 0) \tag{5.5}$$

where $K_b$ and $K_c$ are the S.I.F's at the tip "$b$" and "$c$"of the crack respectively. Substituting from (5.3) in (5.4) and (5.5) respectively and evaluating the limits we get

$$K_b = \frac{2}{?}\left[ n(b) \frac{2}{?^2} \int_b^c \frac{x?(x)\,P_1(x)}{b^2 - x^2}\,dx \right] \tag{5.6}$$

$$K_c = -\frac{2}{?}\left[ n(c) \frac{2}{?^2} \int_b^c \frac{x?(x)\,p_1(x)}{c^2 - x^2}\,dx - t_1(c) \right] \tag{5.7}$$

where

$$n(x) = \left[ 2x(c^2 + b^2) \right]^{-\frac{1}{2}} \tag{5.8}$$

and

$$t_1(c) = \lim_{x \to c^+} \sqrt{x - c} \int_c^\infty \frac{t u_0'(t)}{t^2 - x^2}\,dt \tag{5.9}$$

**6. Special Cases**

To be more sure of the analysis done, we take up two special cases in which we determine the stress intensity factors at the crack tips ($b$, 0) and ($c$, 0) and then report on the closed form expressions for crack shape.

**Case-I**

In the first case we assume that, the wedge shape function $u_0(x)$, temperature distribution $\theta(x)$ and the flux given in $b < x < c$ to be constant, i.e.,

$$\begin{aligned} &u_0(x) = u_0 \text{ (Constant)} \\ &\theta(x) = \theta_0 \text{ (Constant)} \\ &f(x) = f_0 \text{ (Constant)} \quad \text{for } b < x < c \end{aligned} \tag{6.1}$$

Now to evaluate $h_0(t)$ of (4.5), we need $p(x)$, which from (4.7) on using (6.1) it is reduced to:

$$P(x) = ?f_0 \tag{6.2}$$

on substituting this value of $P(x)$ in (4.5) and solving the integral, we get

$$h_0(t) = \frac{1}{?\text{q}^2?\text{q}(t)}\left[\frac{-?‡}{4} fo\,(2t^2 - b^2 - c^2) + D\right] \qquad (6.3)$$

where the constant $D$ can be easily obtained by integrating (6.3) through (4.3), as

$$D = \frac{1}{f}\left[?.^2 c?(b) + \frac{?\text{i}}{f} fo\,\{(b^2 + c^2)F - 2c^2 E)\right] \qquad (6.4)$$

where F and E are complete elliptic integrals of the first and of the second kind respectively given by

$$F = F\,(\frac{?‡}{2},\ ?^{a}_{0}),\ \mathrm{E} = \mathrm{E}\,(\frac{?‡}{2},\ ?^{a}_{0}) \qquad (6.5)$$

and $?‡_0$ is given by

$$?‡_0 = \frac{?\text{t}}{2}, ?\text{ý}_0 \qquad (6.6)$$

Now to get the crack shape we need $h_1$ (t), which is firstly needed to obtain $p_1$(x) on using (6.1) in (4.11), given as

$$p_1(x) = \lambda T\,(x,0) \qquad (6.7)$$

with

$$\lambda = -\frac{(3?‡^2 - 4)?\tilde{A}_t}{2(?«^2 - 1)} \qquad (6.8)$$

$T$ ($x$.0) is obtained by integrating $h_0$ (t) from (6.3) through (5.1) and obtained as

$$T(x, 0) = \frac{c}{?‡^2}\left[\frac{?\text{S}}{4} fo\{2c^2 E(\Phi, ?Ø_0) - (b^2 + c^2)F(\Phi, ?‡_0)\right] \qquad (6.9)$$

where:

$$= Sin^{-1}\sqrt{\frac{c^2+x^2}{c^2-b^2}} \qquad b<x<\mathrm{c} \qquad (6.10)$$

Substituting for $p_1(x)$ from (6.7) in (4.10) we get

$$h_0(t) = \frac{1}{?^2?\,(t)}\left[?\chi\int_b^c \frac{x?(x)}{t^2-x^2}+T\,(x,0)dx+D_1\right] \qquad (6.11)$$

where the constant $D_1$ is obtained by integrating (6.11) through (4.9). The stress intensity factors at this stages are

$$\Delta_0(b) = \frac{1}{?^2}\left[?\phi\int_b^c \frac{x?\,(x)}{t^2-x^2}+T\,(x,0)dx+D_1\right] \qquad (6.12a)$$

$$\Delta_0(c) = \frac{1}{?\ddagger^2}\left[?A\int_b^c x\frac{x?\,(x)}{t^2-x^2}+T\,(x,0)dx+D_1\right] \qquad (6.12b)$$

with $T\,(x,0)$ in (6.9). Now the crack shape can be easily determined by integrating (6.11) through (5.2). For numerical purposes the integrals in (6.12) can be solved directly by applying Simpson's quadrature formula.

**Case-II**

As in case-I we keep the wedge shape function $u_0(x)$ and the temperature distribution $\theta(x)$ as constant and assume that there is no flux $f(x)$ in $b<x<c$, *i.e.*, the surface of the crack is kept insulated, therefore we have the boundary conditions as

$$\begin{aligned} u_0(x) &= u_0\ \text{(Constant)} \\ \theta(x) &= \theta_0\ \text{(Constant)} \\ f(x) &= f_0\ \text{(Constant)} \qquad b<x<c \end{aligned} \qquad (6.13)$$

If we proceed as in case (I) we can easily calculate all the other physical quantities such as normal stress components, crack shape, stress intensity factors, and etc.

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