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A Minimum Cost Relaxation Model for Infeasible Flow Networks

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Abstract

In many real systems, it happens that the existing flow network become inconsistent with the new applications or inputs. This means that some of the applicable structural characteristics have been changed so that the flow network has become infeasible or, in other words, obsolete. Therefore, it has to be adjusted to new applications. It is well known how to use a maximum flow algorithm to determine when a flow network is infeasible, but less known is how to adjust the structural data such that the network becomes feasible while the incurred adjustment cost is minimal. This paper considers an infeasible flow network G= (V, A) in which supplies/demands, arc capacities and flow lower bounds are liable to relax. A minimum cost relaxation model for canceling most positive cuts is constructed. Analyzing the model shows that, in order to make the network feasible, it is sufficient to adjust only one component of the structural data. According to this result, a polynomial time algorithm is developed to cancel all positive cuts and convert the infeasible flow network to a feasible one.

Keywords: *Network flow, cut, minimum cost flow, time order.*

1. Introduction

Consider a flow network G= (V, A) with supplies/ demand b, arcs capacity u and flow lower bounds l. The supplies / demands are such that $\sum_{i \in v} b_i = 0$. A flow $x = (x_{ij})$ is called *feasible* if it satisfies

$$\sum_j x_{ij} - \sum_k x_{ki} = b_i \text{ , for all } i \in V \tag{1.a}$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \text{ , for all } (i, j) \in A \tag{1.b}$$

The constraints (1.a) and (1.b) are called *conservative (or mass balance) and bound conditions* respectively. If a flow networks admits a feasible flow then it is called a *feasible flow Network* (FFN), otherwise it is an *Infeasible flow Network* (IFN). Feasibility of a given flow network can be checked by solving a maximum flow problem defined on the related extended network as follows:

Add two artificial nodes s and t as source and sink, respectively, to V; for each supply node $i\,(b_i > 0)$ add arcs (s, i) with $l_{si} = 0$ and $u_{si} = b_i$. Similarly for each demand node $i\,(b_i < 0)$ add arcs (i, t) with $l_{it} = 0$ and $u_{it} = -b_i$. Now if the solution of the maximum flow problem on this extended network saturates the arcs leaving s or entering t then the original flow network is feasible, since if we omit the components of this solution corresponding to the added artificial arcs a feasible flow for the original network is obtained. Therefore, if the maximum flow does not saturate the mentioned arcs we conclude that the flow network is infeasible and its structural data have to be modified. We are going to find a method that converts an IFN to a FFN with the least modifying cost.

2. Positive cut canceling method

Network flow infeasibility can also be diagnosed by means of cut values. If S is a non- trivial subset of V (*i.e.* $S \neq$?‡ $S \neq V$) a *cut* $(S, \bar{S})$ is defined as

$$(S, \bar{S}) = \{\ (i, j) \in A \mid i \in S,\ \ j \notin S\ \}.$$

The cut's capacity and value are defined as

$$u\,(S, \bar{S}) = \sum_{(i,j)\in(s,\bar{s})} u_{ij} - \sum_{(i,j)\in(\bar{s},s)} l_{ij}$$

and

$$V\,(S, \bar{S}) = b\,(S) - u\,(S, \bar{S}),$$

respectively, where $b(S) = \sum_{i \in S} b_i$, [See (Ahuja *et al.*, 1993)].

It has been proved (Ahuja *et al.*, 1993 and Hoffman, 1960) that a flow network is infeasible if and only if there exists a cut $(S,\bar{S})$ such that

$$V(S,\bar{S}) > 0 \tag{2}$$

such a subset is called a *witness of infeasibility* (Aggarwal *et al.*, 1998). The superiority of this method, regarding the maximum flow method, is that it establishes a practical way to convert an IFN to FFN.

In an IFN, depending on its structure, it may be possible to adjust or modify all of the structural data. That is the supplies / demands, capacities and flow lower bounds can (or have to) be modified. In other cases modifying only some of the structural data, i.e. only flow bounds, may be an obligation. In this paper, we consider both of these cases.

In an IFN, in order to achieve feasibility, the constraint (2) has to be canceled for all positive cuts. This can be done only by adjusting the structural data. Denote the cost of one unit change in supply or demand of node i by C_i for all $i \in V$, and cost of changing one unit in flow bounds (capacity or lower bound) of arc $(i.j)$ by C_{ij} for all (i,j) $\in A$.

Now suppose $(S,\bar{S})$ is a positive cut in the flow network. If the amounts of change in b_i's, l_{ij}'s and u_{ij}'s are denoted by d_i, p_{ij} and q_{ij} respectively, then the *Minimum cost Relaxation Problem* (MCRP) for canceling the cut $(S,\bar{S})$ is

$$\text{Min} \sum_{i\in s} c_i d_i + \sum_{(i,j)\in(\bar{s},s)} c_{ij}\, p_{ij} + \sum_{(i,j)\in(s.\bar{s})} c_{ij} q_{ij} \tag{3.a}$$

s.t.

$$\sum_{i\in s}(b_i - d_i) + \sum_{(i,j)\in(\bar{s},s)}(l_{ij} - p_{ij}) - \sum_{(i,j)\in(s,\bar{s})}(u_{ij} + q_{ij}) \leq \tag{3.b}$$

$$d_i \geq 0\ ,\ \ i \in S \tag{3.c}$$

$$p_{ij} \geq 0\ ,\ \ (i,j) \in (\bar{S},S) \tag{3.d}$$

$$q_{ij} \geq 0\ ,\ (i.j) \in (S,\bar{S}) \tag{3.e}$$

in the above problem, we have assumed that the costs of modifying the flow lower and upper bounds for each arc are equal. As it is seen, the minimization model has been constructed based on all data modification. If only the flow bounds are permitted to change, the problem will be modeled as

$$\text{Min} \sum_{(i,j)\in(\bar{s}/s)} c_{ij}\, p_{ij} + \sum_{(i,j)\in(s,\bar{s})} c_{ij} q_{ij} \tag{4}$$

s.t

$$\sum_{i\in s} b_i + \sum_{(i,j)\in(\bar{s}/s)} (l_{ij} - p_{ij}) - \sum_{(i,j)\in(s/\bar{s})} (u_{ij} + q_{ij}) \le 0$$

$$p_{ij} \ge o \ , \ (i,j) \in (\bar{S}, S)$$

$$q_{ij} \ge o \ , \ (i,j) \in (S, \bar{S})$$

now define $C' =$ min. $\{ C_i \,/\, i \in S \}$, $C'' =$ min $\{ C_{ij} \,/\, (i,j) \in S\bar{S} \}$ and $\bar{C} =$ min. $\{ C', C'' \}$, where $S\bar{S} = (S, \bar{S}) \cup (\bar{S}, S)$.

Theorem 1. The optimal solution of MCRP consists of V$(S,\bar{S})$ for the variable corresponding to $\bar{C}$ and zero for all other variables.

Proof. Since (3.b) can be rewritten as

$$\sum_{i\in S} d_i + \sum_{(i,j)\in(\bar{S},S)} p_{ij} + \sum_{(i,j)\in(S,\bar{S})} q_{ij} \ge \text{V}\,(S,\bar{S}),$$

the dual of problem (3) is

$$\text{Maximize V}\,(S,\ \bar{S})\ \text{w}$$

s.t.

$$\text{w} \le C_i, i \in S$$

$$\text{w} \le C_{ij}, (i,j) \in S\bar{S}$$

$$\text{w} \ge 0$$

the optimal solution of this problem is w* = $\bar{C}$.

According to the definition of $\overline{C}$, either $\overline{C} = C_k$ for some $K \in S$, or $\overline{C} = C_{kl}$ for some $(k,l) \in S\overline{S}$. We call d_k or p_{kl} or q_{kl}, accordingly, the variable corresponding to $\overline{C}$. Now set the value of this variable equal to V $(S,\overline{S})$ and the values of all other variables equal to zero. The objective function then equals to $\overline{C}$ V $(S,\overline{S})$ and the constraint (3.b) becomes binding. Therefore we get two primal and dual feasible solutions for which the value of both objective function is equal to $\overline{C}$ V$(S,\overline{S})$. According to the weak duality theorem, both solutions are optimal.

This Theorem may be proved by a simpler argument. Note that all the variables and parameters in the problem are nonnegative. So setting all variables equal to zero, but the one with the smallest coefficient, which is set equal to V$(S,\overline{S})$, will give a feasible solution with the minimum objective value. •

Corollary1. If $\overline{C}$ = min $\{C_{ij} / \ (i,j) \in S\overline{S}\}$, then the optimal solution of (4) consists of V $(S,\overline{S})$ for the variable corresponding to $\overline{C}$ and zero for other variables. •

2.1 Minimum cost algorithm.

The above Theorem and corollary provide a very simple and efficient method for canceling a positive cut. It is sufficient to modify only one component of the structural data of the flow network by v$(S,\overline{S})$. In Fig.1, we present a high-level descriptive algorithm that can be used for both problems (3) and (4).

In the algorithm the procedure *mostpositive-cut* finds a most positive cut $(S,\overline{S})$ and returns its value if there exists such a cut; otherwise, it returns a negative value. A straightforward way to compute the most positive cut is to apply a minimum flow algorithm to the network.

Minimum changing cost algorithm

begin
while CV= *mostpositive- cut* (S) > 0 **do**
begin
$\bar{S}$ = V\ S,
C1= min { $C_i : i \in S$ } ; k = index of the minimum;
C2= min { $C_{ij} : (i, j) \in (S, \bar{S})$ }; (p, q) = index of the minimum;
C3= min { $C_{ij} : (i, j) \in (\bar{S}, S)$ } ; (r, y) = index of the;
C4= min { C_2, C_3 }
if $C_1 <= C_4$ **then** $b_k = b_k - CV$
else if $C_2 <= C_3$ **then** $u_{pq} = u_{pq} + CV$
else $l_{ry} = l_{ry} - CV$
end
end

Fig. 1- Cut canceling algorithm

If the supply/demand values are not allowed to change, then in the algorithm of Fig. 1 only C2 and C3 are computed and the first if-then part of the final statement is deleted

2.2 Minimum magnitude algorithm.

In the systems where modification cost is of minor importance, the objective can be set to minimize the largest changing value. In such a case, for a positive cut ($S, \bar{S}$) define

$$D = \{ d_i / i \in S \},$$
$$P = \{ p_{ij} / (i, j) \in (S, \bar{S}) \},$$

$$Q = \{ q_{ij} \,/\, (i,j) \in (S,\bar{S}) \}$$

now the cut will be canceled by solving the following problem

Minimize Max. $D \cup P \cup Q$

(5)

s.t.

$$\sum_{i \in s}(b_i - d_i) + \sum_{(i,j)\in(\bar{s},s)}(l_{ij} - p_{ij}) - \sum_{(i,j)\in(s,\bar{s})}(u_{ij} + q_{ij}) \le o$$

$$d_i \ge 0 \;,\; i \in S$$

$$p_{ij} \ge 0 \;,\; (i,j) \in (\bar{S},S)$$

$$q_{ij} \ge 0 \;,\; (i,j) \in (S,\bar{S})$$

In cases where the supplies/ demands are not permitted to change the problem will be

Minimize Max. P $\cup$ Q

(6)

s.t.

$$\sum_{i \in s} b_i + \sum_{(i,j)\in(\bar{s},s)}(l_{ij} - p_{ij}) - \sum_{(i,j)\in(s,\bar{s})}(u_{ij} + q_{ij}) \le 0$$

$$p_{ij} \ge 0 \;,\; (i,j) \in (\bar{S},S)$$

$$q_{ij} \ge 0 \;,\; (i,j) \in (\bar{S},S)$$

Theorem 2. Each component of the optimal solution of problem (5) is equal to V $(S,\bar{S})$/k, where k = $| D \cup P \cup Q |$.

Proof. If Max. $D \cup P \cup Q$ is denoted by z, then the problem (5) can be stated as

Minimize z

(7)

s.t.

$$z - d_i \geq 0,\ i \in S$$
$$z - p_{ij} \geq 0,\ (i,j) \in (\bar{S}, S)$$
$$z - q_{ij} \geq 0\ ,\ (i,j) \in (S, \bar{S})$$
$$\sum_{i \in s} d_i + \sum_{(i,j) \in (\bar{s}, s)} p_{ij} + \sum_{(i,j) \in (s, \bar{s})} q_{ij} \geq V(S, \bar{S})$$
$$d_i \geq 0\ \ i \in S$$
$$p_{ij} \geq 0\ ,\ (i,j) \in (\bar{S}, S)$$
$$q_{ij} \geq 0\ ,\ (i,j) \in (S, \bar{S})$$

The dual problem of (7) is as follows

Maximize v($s, \bar{s}$) w_{k+1}
(8)
s.t.

$$\sum_{i=1}^{k} w_i \leq 1$$
$$w_{k+1} - w_i \leq 0,\ i = 1,2,..K$$
$$w_i \geq 0\ ,\ i = 1,2,..,.K+1$$

now consider a (k+1)–vector W^* with all entries equal to 1/k. W^* is a feasible solution of the dual problem (8). Set

$$d_i^* = \text{v}\,(S, \bar{S})\,/\,\text{k}, i \in S\,,$$
$$p_{ij}^* = \text{v}\,(S, \bar{S})/\text{k},\ (i,j) \in (\bar{S}, S)\,,$$
$$q_{ij}^* = \text{v}\,(S, \bar{S})/\text{k},\ (i,j) \in (S, \bar{S})\,,$$
$$z^* = \text{v}\,(S, \bar{S})/\text{k}.$$

These values constitute a feasible solution for problem (7). The objective functions of the primal problem (7) and the dual problem (8) have the same value for these feasible solutions. Therefore, both are optimal.

According to this theorem, the counter part to the minimum cost algorithm is as follows

Minimum magnitude algorithm

begin
while CV=*mostpositive – cut* (S) > 0 **do**
begin
$\bar{S} = V \setminus S$
$k = \left| S \cup (S, \bar{S}) \cup (\bar{S}, S) \right|$;
$b_i = b_i - CV/k,$ for each $i \in S$;
$l_{ij} = l_{ij} - CV/K,$ for each $(i, j) \in (\bar{S}, S)$;
$u_{ij} = u_{ij} + CU/K,$ for each $(i, j) \in (\bar{S}, \bar{S})$;
end
end

Fig 2. Minimum magnitude algorithm.

For the systems where b_i's are not allowed to change, the updating instruction of b_i is deleted from the above algorithm.

Theorem 3. The positive-cut canceling algorithm converges to a feasible flow network in at most O([nB+ mL] MF (n, m)) time where B= $\max_{i \in v}. \ b_i$, L= $\max_{(i,j) \in A} l_{ij}$ and MF (n, m) is time to compute a maximum flow.

Proof. Computing the most positive cut is the most important step at each iteration. McCormick *et al.,* proved that a most positive cut can be computed in O(Mc (m, n)) time, where Mc (m, n) is the running time of the fastest minimum cut algorithm in a network with m arcs and n nodes (McCormick and Ervolina, 1994).

Clearly Mc (m, n) is less than or equal to the time of computing maximum flow. The fastest known strongly polynomial maximum flow algorithm is due to Goldberg and Tarjan (Goldberg and Tarjan, 1988) with MF (m, n) = O(nmlog (n^2/m) time order.

In the algorithm, if the procedure *mpositive-cut* (S) retunes a negative value the algorithm terminates and the resulting flow network is feasible, since it contains no positive cut. Otherwise a most positive cut (i.e. S) with positive value CV is returned.

In the next steps of the algorithms the networks data are modified in at most o(n+m) time, If after the modification l_{ij} the data have the values l', u' and b' then $l'_{ij} \leq l_{ij}$, $U'_{ij} \geq U_{ij}$ for each $(i, j) \in A$ and $b'_i < b_i$ for each $i \in V$ and at least one of these inequalities is strict. This means the next most positive cut (to be found in the next iteration, if it exists) has a value strictly less than current CV. By the assumption of integrality of the data, this decrease is at least one unit.

Now consider that

$$\mathrm{V}\,(S,\bar{S}) \leq \sum_{i \in s} b_i + \sum_{(i,j) \in (\underline{S},S)} l_{ij} \leq \mathrm{nB+\ mL}$$

Therefore the maximum number of iterations in the algorithms (while loop) will nB+mL and, hence the overall time order of both algorithms is O ((B+mL) nmlog (n^2/m])) •

3. Conclusion

In this article, we considered flow networks that, due to change of application properties were impractical or infeasible. In order to convert such networks to feasible ones, their structural properties, i.e. arcs' capacity, flow lower bounds and supply/demand values, have to be adjusted. A mathematical model for minimization of total adjusting cost was constructed and solved. Analyzing the model showed that the optimal solution included only one component's change in the network and the amount of change was equal to the value of a positive cut.

Based on the above result, a polynomial time algorithm was presented to find and apply the optimal solution to the network.

References

Aggarwal Charu C., Ravndra K. Ahuja, Jianxiu Hao, James B. Orlin, (1998) *Diagnosing infeasibilities in network flow problems,* Mathematical Programming, **81**, 263-280.

Ahuja R.K., Magnanti T.L.and Orlin J.B,(1993) Network flows: Theory, algorithms and applications, Prentice Hall, New York, NY.

Goldberg, A.V. and Tarjan, R.E. (1988) *A new approach to the maximum flow problem,* J. ACM, **53**(4), 921-940.

Hoffman, A.J. (1960) *Some recent applications of the theory of linear inequalities to extremal Combinatorial analysis*, R. Bellman and M. Hall, Jr., eds., Proceeding of symposia in Applied Mathematics, American Maths. Society, 113-127.

McCormick S. T. and Ervolina T. R., (1994) *Computing maximum mean cuts*, Discrete Applied Mathematics, **52**(1), 53-70.