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Boundary Behavior of Biharmonic Functions in the Unit Disk

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Abstract

We shall first introduce a new biharmonic Poisson kernel for the unit disk in the complex plane, and then proceed to study the boundary behavior of the potentials (biharmonic functions) related to this kernel function.

Keywords: *Biharmonic function, Biharmonic Poisson kernel, Biharmonic Green function, Nontangential limit.*

0 - Introduction

We denote by D the unit disk and by T the unit circle in the complex plane. A locally integrable function u defined on the unit disk D is said to be biharmonic provided that $\Delta^2 u = 0$; here the symbol Δ stands for the Laplacian, and the mentioned equation is interpreted in the sense of distributions. The biharmonic Green function for the operator Δ^2 in the unit disk is the function

$$\Gamma(z,\zeta) = |z-\zeta|^2 \log\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right|^2 + (1-|z|^2)(1-|\zeta|^2), \qquad (z,\zeta)\in D\times D.$$

We mention that for fixed $\zeta \in D$, the biharmonic Green function solves the following boundary value problem

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$$\begin{cases} \Delta_z^2 \Gamma(z,\zeta) = \delta_\zeta(z), & z \in D, \\ \Gamma(z,\zeta) = 0, & z \in T, \\ \partial_{n(z)} \Gamma(z,\zeta) = 0, & z \in T, \end{cases}$$

where the notation $\partial_{n(z)}$ denotes the inward normal derivative (in the sense of distributions) with respect to the variable $z \in T$.
We define the *biharmonic Poisson kernel* $F(\zeta, z)$ for the unit disk by the following relation

$$F(\zeta,z) = -\frac{1}{2}\partial_{n(\zeta)}\Delta_\zeta \Gamma(z,\zeta) = \frac{1}{2}\left\{\frac{(1-|z|^2)^2}{|1-\bar{z}\zeta|^2} + \frac{(1-|z|^2)^3}{|1-\bar{z}\zeta|^4}\right\}, \quad (\zeta,z) \in T \times D.$$

We shall see that for fixed ζ on the boundary of the region, F (ζ, z) is a biharmonic function in the z variable, moreover, F (ζ, z) enjoys the properties of a kernel function. Using this kernel function, we consider its relevant potentials to generate a class of biharmonic functions in the unit disk. More precisely, given $f \in L^1(T)$, we define the *F-integral* of f by

$$u(z) = F[f](z) = \int_T F(\zeta,z) f(\zeta) d\sigma(\zeta), \qquad z \in D,$$

where $d\sigma(\zeta) = (2\pi)^{-1} d\theta$ for $\zeta = e^{i\theta}$, stands for the normalized arc length measure on the unit circle.

We intend to study the boundary behavior of the biharmonic function $u = F[f]$ when the variable approaches the boundary of the unit disk. We shall see that Fatou's theorem concerning the existence almost everywhere of nontangential limits is valid in this situation, so that from this perspective, the biharmonic Poisson kernel resembles the usual (harmonic) Poisson kernel.

It is desirable to have some words on the origin of the biharmonic Poisson kernel $F(\zeta, z)$. Indeed, it was appeared in a Riesz-type representation formula found by the current author and Hedenmalm in

[**AH**]. To give a brief account on the origin of our biharmonic Poisson kernel, we need some more notations.

We write $dA(z) = \pi^{-1} dxdy$ for the normalized Lebesgue area measure on the unit disk. We also write u_r for the dilation of u by r, $0 < r < 1$: $u_r(z) = u(rz)$. In this way, we may think of u_r as a function on the unit circle T.

For a C^{∞}-smooth function u on the closed unit disk $\overline{D}$, the Poisson-Jensen formula, which is an immediate consequence of Green's formula, represents u as

$$u(z) = \int_D G(z,\zeta)\Delta u(\zeta)\, dA(\zeta) + \int_T P(z,\zeta) u(\zeta)\, d\sigma(\zeta), \quad z \in D. \qquad (0-1)$$

Here $G(z,\zeta)$ stands for the Green function for Laplace operator in the unit disk

$$G(z,\zeta) = \log\left|\frac{z-\zeta}{1-\bar{z}\zeta}\right|^2, \qquad (z,\zeta) \in D \times D,$$

and $P(z,\zeta)$ denotes the Poisson kernel for the unit disk:

$$P(z,\zeta) = \frac{1-|z|^2}{|z-\zeta|^2}, \qquad (z,\zeta) \in D \times T.$$

A function $u : D \to [-\infty, +\infty[$ is said to be *subharmonic* if it is upper semicontinuous and satisfies the sub-mean value inequality

$$u(z) \le \int_T u(z + r\zeta)\, d\sigma(\zeta), \qquad z \in D, \quad 0 \le r < 1 - |z|.$$

The Poisson-Jensen representation formula (0–1) is valid in the context of subharmonic functions u under the growth assumption

$$\sup_{0 \le r < 1} \int_T u^+(rz)\, d\sigma(z) < +\infty,$$

where the superscript $+$ stands for the positive part of the function u. It follows under these assumptions that the Poisson-Jensen representation formula (0–1) generalizes to

$$u(z) = \int_D G(z,\zeta)\, d\mu(\zeta) + \int_T P(z,\zeta)\, d\nu(\zeta), \qquad z \in D,$$

where μ is a positive Borel measure on D with

$$\int_D (1-|z|^2)^2 \, d\mu(z) < +\infty.$$

and ν is a finite real-valued Borel measure on T. The measure μ corresponds to the Laplacian Δu, and ν corresponds to the boundary values of u.

Let u be a locally integrable real-valued function on the unit disk. The function u is said to be *super-biharmonic* provided that $\Delta^2 u$ is a locally finite positive Borel measure on D; in other words, $\Delta^2 u \geq 0$ in the sense of distributions.

We want to find a representation formula, analogous to (0–1), for a class of super-biharmonic functions. Let us first review the smooth case. Suppose that u is a C^∞-smooth function on $\overline{D}$. Applying Green's formula twice we obtain the representation

$$\begin{aligned} u(z) = {} & \int_D \Gamma(\zeta, z)\Delta^2 u(\zeta)\, dA(\zeta) - \frac{1}{2}\int_T \partial_{n(\zeta)}\big(\Delta_\zeta \Gamma(\zeta, z)\big) u(\zeta)\, ds(\zeta) \\ & + \frac{1}{2}\int_T \big(\Delta_\zeta \Gamma(\zeta, z)\big)\partial_{n(\zeta)} u(\zeta)\, ds(\zeta), \quad z \in D. \end{aligned} \tag{0 – 2}$$

A computation shows that

$$\Delta_\zeta \Gamma(\zeta, z) = G(\zeta, z) + H(\zeta, z), \qquad (\zeta, z) \in D \times D,$$

where

$$H(\zeta,z)=(1-|z|^2)\frac{1-|z\zeta|^2}{|1-\bar{z}\zeta|^2},\qquad (\zeta,z)\in \overline{D}\times D. \qquad (0-3)$$

We shall refer to $H(\zeta,z)$ as the *harmonic compensator*; it is harmonic in its first argument and is biharmonic in its second argument. Observe that $H(\zeta,z)$ is not symmetric in its arguments. Another computation shows that the function

$$F(\zeta,z)=-\frac{1}{2}\partial_{n(\zeta)}\Delta_{\zeta}\Gamma(\zeta,z),\qquad (\zeta,z)\in T\times D,$$

has the form

$$F(\zeta,z)=\frac{1}{2}\left\{\frac{(1-|z|^2)^2}{|1-\bar{z}\zeta|^2}+\frac{(1-|z|^2)^3}{|1-\bar{z}\zeta|^4}\right\},\qquad (\zeta,z)\in T\times D. \qquad (0-4)$$

Being biharmonic in its second argument, the function $F(\zeta,z)$ will be referred to as the *biharmonic Poisson kernel*. Note that in terms of the above kernels, (0–2) assumes the form

$$u(z)=\int_D \Gamma(\zeta,z)\Delta^2 u(\zeta)\,dA(\zeta)-\frac{1}{2}\int_T F(\zeta,z)\,u(\zeta)\,ds(\zeta)$$
$$+\frac{1}{2}\int_T H(\zeta,z)\,\partial_{n(\zeta)}u(\zeta)\,ds(\zeta),\quad z\in D.$$

For a possibly non-smooth function u, it is natural to ask when we have the representation formula

$$u(z)=\int_D \Gamma(\zeta,z)\,d\mu(\zeta)+\int_T F(\zeta,z)\,d\nu(\zeta)+\int_T H(\zeta,z)\,d\lambda(\zeta),\qquad z\in D,$$

where ν and λ are two real-valued finite Borel measures on T and

?‡is a positive Borel measure on D with

$$\int_D (1-|z|^2)^2 \; d?‡(z) < +\infty.$$

Clearly, u has to be super-biharmonic. Moreover, it can be seen that it meets the growth conditions

$$\sup_{0 \le r < 1} \int_T u^+(rz) \; d?‡(z) < +\infty, \tag{A}$$

which assures that $u_r d?‡$ has at least one weak-star cluster point ?‡as $r \to 1$, and

$$\sup_{0 \le r < 1} \frac{1}{1-r} \int_T \big(u - F[?]\big)^-(rz) \;\; d?‡(z) < +\infty, \tag{B}$$

where

$$F[?](z) = \int_T F(?, z) \; d?(?), \qquad z \in D. \tag{0 – 5}$$

It is a consequence of the second assumption (**B**) that the measure $d?‡$ obtained from this limit process is unique: $u_r \; d?, \to d?‡$ weak-star, as $r \to 1$. It now makes sense to ask whether the conditions (**A**) and (**B**) characterize the super-biharmonic functions u on D having the above representation. The following theorem was proved in [**AH**].

Theorem (Abkar-Hedenmalm). For a locally integrable function u on D, the following two conditions are equivalent:

(a) u has the representation

$$u(z) = \int_D \Gamma(?, z) \; d?(?) + \int_T F(?, z) \;\; d?(?) + \int_T H(?, z) \;\; d?(?‡), \quad z \in D,$$

where ?‡is a positive Borel measure on D with

$$\int_D (1-|z|^2)^2 \; d?‡(z) < +\infty,$$

and ?, ?‡ are two finite real-valued Borel measures on T.

(b) u is super-biharmonic on D, and meets the growth conditions **(A)** and **(B)**.

As was pointed out earlier, the above theorem is the place where the biharmonic Poisson kernel first appeared. In the next section we shall return to the main line of development by considering the potentials associated to the biharmonic Poisson kernel.

1. The Biharmonic Extension

In this section we study the kernel function $F(?, z)$, defined by (0–4), in more detail. We denote by $C(T)$ the space of continuous functions on the unit circle T. It turns out that every $f \in C(T)$ has a biharmonic extension Bf to the closed unit disk; moreover, the function Bf is continuous on $\overline{D}$, and $Bf(rz) \rightarrow f(z)$ uniformly, as $r \rightarrow 1$. We use this fact to prove that the measure ?‡ on T which is obtained as a weak-star limit of $u_r d$?‡ has some kind of uniqueness property. We first note that

$$F(?, z) > 0, \qquad (?, z) \in T \times D, \tag{1-1}$$

and that

$$\int_T F(?, z)\, d?(?) = 1, \qquad z \in D. \tag{1-2}$$

The equality (1–2) follows from the following simple calculation:

$$\int_T F(\zeta,z)\, d\sigma(\zeta) = \frac{1}{2}\int_T \frac{(1-|z|^2)^2}{|1-\bar{z}\zeta|^2}\, d\sigma(\zeta) + \int_T \frac{(1-|z|^2)^3}{|1-\bar{z}\zeta|^4}\, d\sigma(\zeta)$$
$$= \frac{1}{2}\Big((1-|z|^2)+(1+|z|^2)\Big)$$
$$=1.$$

For $\zeta \in T$, we let $I(\zeta)$ be the arc on the unit circle with center ζ and length $\delta > 0$. For $z \in D\setminus\{0\}$, we write $z^* = z/|z|$ which belongs to the unit circle. It follows from (0–4) that $F(\zeta,z)\to 0$ uniformly, as $|z|\to 1$ and $z^* \in T\setminus I(\zeta)$. Note that the Poisson kernel for the unit disk satisfies the relations (1–1), (1–2) and this last property as well.

Let $f \in L^1(T)$. Recall the F-integral of f given by

$$F[f](z) = \int_T F(\zeta,z)\, f(\zeta)\, d\sigma(\zeta), \qquad z \in D.$$

Since $F(\zeta,z)$, for fixed $\zeta \in T$, is biharmonic, it follows that $F[f]$ is biharmonic in D. The following proposition shows that the F-integrals of continuous functions behave well near the boundary of the unit disk.

Proposition 1.1. Let $f \in C(T)$ and $z \in T$. Define the biharmonic extension of f to the unit disk by

$$(Bf)(z) = \begin{cases} f(z), & z \in T, \\ F[f](z), & z \in D. \end{cases}$$

Then Bf is continuous on $\overline{D}$.

Proof. For a subset E of the complex plane, we write

$$\|f\|_E = \sup\{\,|f(z)|: \quad z \in E\}.$$

Assume that $f \in C(T)$ and that $z \in D$. It follows from (1–1) and (1–2) that

$$\left|F[f](z)\right| = \left|\int_T F(?,z) f(?)\, d?(?)\right| \le \|f\|_T .$$

Hence

$$\|Bf\|_{\overline{D}} = \|f\|_T, \qquad f \in C(T). \qquad (1-3)$$

Put $g_n(z) = z^n$ and compute

$$(Bg_n)(z) = \int_T F(?,z) g_n(?)\, d?(?) = \int_T F(?,z) ?^n\, d?(?)$$
$$= z^n \left(1 + \frac{n}{2}(1-|z|^2)\right), \qquad z \in D.$$

Let $p(z) = \sum_{n=-k}^{k} c_n z^n$ be a trigonometric polynomial on the unit circle. It follows that for every such polynomial p, we have

$$(Bp)(rz) = \sum_{-k}^{k} c_n r^{|n|} z^n \left(1 + \frac{n}{2}(1-|z|^2)\right), \qquad z \in D.$$

This implies that Bp is continuous on $\overline{D}$. Since the trigonometric polynomials are dense in $C(T)$, we can assume that $\{p_n\}_{n\ge 1}$ is a sequence of such polynomials on T such that $\|p_n - f\|_T \to 0$ as $n \to \infty$. It follows from (1–3) that

$$\|Bp_n - Bf\|_{\overline{D}} = \|B(p_n - f)\|_D = \|B(p_n - f)\|_T$$
$$= \|p_n - f\|_T \to 0 \qquad as \quad n \to \infty.$$

Hence Bp_n converges uniformly to Bf on $\overline{D}$. Since each Bp_n is continuous on $\overline{D}$, we see that Bf is a continuous function on the closure of the unit disk.

2. The Boundary Behavior of Potentials

Let $P(z,?¼$ denote the Poisson kernel for the unit disk, and consider the Poisson integral of $f \in L^1(T)$, that is

$$P[f](z) = \int_T P(z,?‡) f(?‡) \, d?‡(?‡), \qquad z \in D.$$

According to a theorem of Fatou, $P[f]$ has nontangential limits, almost everywhere on the boundary (see [**GA**] or [**RU**]). We now consider the F-integral of f given by $u(z) = F[f](z)$. The main result of this paper is an analog of Fatou's theorem: the function u has nontangential limit almost everywhere on the unit circle.

Let us fix a real number $?ñ > 1$. For $?ó \in T$, we define

$$\Omega_{?‡}(?ù) = \{ z \in D : \quad | z - ?ù| < ?ù(1 - | z |) \}.$$

Now, the time is ripe to state the main result of this paper; a Fatou theorem for biharmonic functions associated to the biharmonic Poisson kernel.

Theorem 2.1. Let $f \in L^1(T)$ and let $u = F[f]$. Then u has nontangential limit for almost every $z \in T$; that is

$$\lim_{\Omega_{?‡}(?\$) \ni z \to ?‡} u(z) = f(?•), \qquad \textit{for almost every} \quad ?‡ \in T.$$

Before we prove the theorem, we need a lemma which is key to the proof of our theorem. Exploiting the notations of Theorem 2.1, we define the nontangential maximal function of u at ?‡ by

$$u^*_{?\bullet}(?‡) = \sup_{z \in \Omega_{?‡}(?)} |u(z)|.$$

For a subset E of the unit circle, the notation $|E|$ stands for the one-dimensional Lebesgue measure of E.

Lemma 2.2. Let $f \in L^1(T)$ and $u = F[f]$. Let u^*_α be the nontangential maximal function of u at $\zeta \in T$. Then for every positive number λ we have

$$\left|\left\{\zeta \in T : \quad u^*_\alpha(\zeta) > \lambda\right\}\right| \le \frac{9+12\alpha}{\lambda}\|f\|_{L^1(T)}.$$

Let us postpone the proof of Lemma 2.2 and manage to deduce Fatou's theorem form this Lemma.

Proof of Theorem 2.1. We can assume that f is real-valued (the same argument can be applied to the real and imaginary parts of f). Put

$$\omega_f(\zeta) = \limsup_{\Omega_\alpha(\zeta) \ni z \to \zeta} \left|u(z) - f(\zeta)\right|.$$

It is clear that ω_f is nonnegative, moreover,

$$\omega_f(\zeta) \le \limsup_{\Omega_\alpha(\zeta) \ni z \to \zeta} \left|u(z)\right| + \left|f(\zeta)\right| \le u^*_\alpha(\zeta) + \left|f(\zeta)\right|.$$

This implies that for every $\varepsilon > 0$ we have

$$\left|\left\{\zeta \in T : \quad \omega_f(\zeta) > \varepsilon\right\}\right| \le \left|\left\{\zeta \in T : \quad u^*_\alpha(\zeta) > \frac{\varepsilon}{2}\right\}\right| + \left|\left\{\zeta \in T : \quad |f(\zeta)| > \frac{\varepsilon}{2}\right\}\right|.$$

According to Lemma 2.2,

$$\left|\left\{\zeta \in T : \quad u^*_\alpha(\zeta) > \frac{\varepsilon}{2}\right\}\right| \le \frac{18+24\alpha}{\varepsilon}\|f\|_{L^1(T)}. \qquad (2-1)$$

On the other hand, by Chebyshev's inequality (see [**GA**] or [**RU**])

$$\left|\left\{? \in T: \ |f(?)| > \frac{?}{2}\right\}\right| \le \frac{2}{?}\|f\|_{L^1(T)}. \qquad (2-2)$$

Combining the relations (2–1) and (2–2), we obtain

$$\left|\left\{? \in T: \ ?_f(?) > ?\right\}\right| \le \frac{20+24?}{?}\|f\|_{L^1(T)}.$$

We now assume that g is a continuous function which approximates the function f in the $L^1(T)$-norm; that is $\|f-g\|_{L^1(T)} < ?^2$. Since g is continuous, we conclude that $?_g(?) = 0$, hence $?_f = ?_{f-g}$. We now apply the above estimate to the function $f-g$ to get

$$\begin{aligned}\left|\left\{? \in T: \ ?_f(?) > ?\right\}\right| &\le \left|\left\{? \in T: \ ?_{f-g}(?) > ?\right\}\right| \\ &= \frac{(20+24?)?^2}{?} \\ &= (20+24?)?,\end{aligned}$$

from which it follows that $?_f = 0$ almost everywhere on the unit circle T. In other words,

$$\lim_{\Omega_?(?)\ni z \to ?} u(z) = f(?), \qquad \textit{for almost every} \quad ? \in T.$$

We now turn to the proof of the lemma.

Proof of Lemma 2.2. Recall the Hardy-Littlewood maximal function of $f \in L^1(T)$ defined by

$$Mf(?) = \sup_{I_?} \frac{1}{|I_?|} \int_{I_?} |f|,$$

where $I_{?‡}$ is an arc centered at $?t \in T$. Our first objective is to show that

$$u^{*}_{?‡}(?) \le (3+4?)Mf(?), \qquad ? \in T. \qquad (2-3)$$

We assume temporarily that (2—3) holds and use the well-known fact that the operator $f \;\alpha\; Mf$ is weak-L^1 (see [**RU**]), meaning that for every ?‡positive

$$\left|\{? \in T: \quad Mf(?) > ?\}\right| \le \frac{3}{?}\|f\|_{L^1(T)},$$

to conclude that

$$\left|\{? \in T: \quad u^{*}_{?‡}(?) > ?\}\right| \le \left|\left\{? \in T: \quad Mf(?) > \frac{?}{3+4?}\right\}\right| \le \frac{9+12?}{?}\|f\|_{L^1(T)}.$$

Hence the lemma follows if we verify that (2–3) holds. To this end, we may assume that $? = 1$. Fix a point $z_0 = r_0 e^{i?_0}$ with the condition that $|?_0| \le ?$. Recall the usual (harmonic) Poisson kernel

$$P_{z_0}(?) = P(z_0, e^{i?}) = \frac{1-r_0^2}{1+r_0^2 - 2r_0\cos(? - ?_0)}, \qquad z_0 = r_0 e^{i?_0} \in D.$$

Since $P_{z_0}(?)$ is a decreasing function of $? \in [0, ?]$, it follows that for $|?_0| < |?| \le ?$ we have

$$\sup\left\{P_{z_0}(t): \quad |?| \le t \le ?\right\} = P_{z_0}(|?|).$$

On the other hand, for $|?| \le |?_0|$, the above supremum is attained when $t = ?_0$, and its value is

$$\frac{1-r_0^2}{1+r_0^2-2r_0}=\frac{1+r_0}{1-r_0}.$$

Let us look at the biharmonic Poisson kernel $F(e^{i\theta}, z_0)$ as a function of θ for fixed $z_0 = r_0 e^{i\theta_0}$. For this, we write

$$F_{z_0}(\theta) = F(e^{i\theta}, z_0), \qquad z_0 = r_0 e^{i\theta_0}.$$

As a matter of fact, there is the following interesting relation between the Poisson kernel and the biharmonic Poisson kernel

$$F(\zeta, z) = \frac{1}{2}(1-|z|^2)\{P(z,\zeta) + P^2(z,\zeta)\}, \qquad (z,\zeta) \in D \times T.$$

We now define

$$\Phi_{z_0}(\delta) = \sup\left\{F_{z_0}(t) : |\delta| \le t \le \pi\right\}.$$

The function $\Phi_{z_0}(\theta)$ is an even function on the interval $-\pi \le \theta \le \pi$, it dominates $F_{z_0}(\theta)$, and it is a decreasing function of $0 \le \theta \le \pi$. Indeed, Φ_{z_0} is the least decreasing majorant of F_{z_0}.

Since $Mf = M(|f|)$, we may assume that $f \ge 0$. Suppose that Φ_{z_0} is an increasing limit of a finite combination of characteristic functions of the intervals $(-\delta_k, \delta_k)$. More precisely, there is a sequence of positive numbers c_k with $\sum c_k \le \|\Phi_{z_0}\|_{L^1(T)}$ such that

$$h_n(\theta) = \sum_{k=1}^{n} c_k \left(\frac{1}{2\delta_k} \chi_{(-\delta_k,\delta_k)}(\theta) \right) \to \Phi_{z_0}(\theta), \quad as \quad n \to \infty.$$

It follows from the monotone convergence theorem that

$$\begin{aligned}\int_T f(?)\Phi_{z_0}(?)\,d? &= \lim_{n\to\infty}\int_T f(?)h_n(?)\,d? \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} c_k \frac{1}{2?_k}\int_{-?_k}^{?_k} f(?)\,d? \\ &\le Mf(1)\sum_{k=1}^{n} c_k \\ &\le Mf(1)\left\|\Phi_{z_0}\right\|_{L^1(T)}.\end{aligned} \qquad (2-4)$$

To have an upper bound for the $L^1(T)$-norm of Φ_{z_0}, we first note that for $|?_0| < |?| \le ?$, we have $\Phi_{z_0}(?) = F_{z_0}(|?|)$ and

$$\Phi_{z_0}(?) = \frac{1}{2}(1-r_0^2)\left\{\frac{1+r_0}{1-r_0} + \frac{(1+r_0)^2}{(1-r_0)^2}\right\}, \qquad |?| \le |?_0|.$$

The second thing we need to know is the following estimate (see **[GM]**):

$$\frac{|?_0|}{1-|z_0|} \le ??, \qquad z_0 \in \Omega_{?}(1).$$

Since Φ_{z_0} is an even function, we can write

$$\left\|\Phi_{z_0}\right\|_{L^1(T)} = 2\int_{|?_0|}^{?}\Phi_{z_0}(?)\frac{d?}{2?} + 2\int_0^{|?_0|}\Phi_{z_0}(?)\frac{d?}{2?}.$$

It follows from the definition of Φ_{z_0} that

$$2\int_0^{|?\ddagger|}\Phi_{z_0}(?)\frac{d?}{2?\bullet} \le \frac{2|?_0|}{2?\bullet}\frac{1}{2}\left(1-r_0^2\right)\left\{\frac{1+r_0}{1-r_0}+\frac{(1+r_0)^2}{(1-r_0)^2}\right\}$$

$$= \frac{|?_0|}{2?Z}\left\{(1+r_0)^2+\frac{(1+r_0)^3}{1-r_0}\right\}$$

$$\le \frac{|?_0|}{2?\%}\left(4+\frac{8}{1-|z_0|}\right)$$

$$\le \frac{2|?_0|}{?}+\frac{4|?_0|}{?(1-|z_0|)}$$

$$\le 2+4?\ddagger$$

This yields

$$\left\|\Phi_{z_0}\right\|_{L^1(T)} \le 1+(2+4?) = 3+4?\ddagger$$

Now we use (2–4) to obtain

$$u(z_0) = \int_T f(?)F_{z_0}(?)\frac{d?}{2?} \le \int_T f(?)\Phi_{z_0}(?)\frac{d?}{2?}$$

$$\le Mf(1)\left\|\Phi_{z_0}\right\|_{L^1(T)}$$

$$\le (3+4?\ddagger)Mf(1).$$

Since z_0 was arbitrarily chosen in $\Omega_{?\ddagger}(1)$, we conclude that

$$u^*_{?\ddagger}(1) \le (3+4?)Mf(1).$$

This completes the proof of Lemma 2.2.

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