More on $\theta$-Compact Spaces

M. Caldas$^1$, S. Jafari$^2$

$^1$Departamento de Matemática Aplicada, Universidade Federal Fluminense, Rua Máximo Santos Braga, 24020-140, Niterói, RJ-Brasil - e-mail: gmamccs@vm.uff.br
$^2$Department of Mathematics and Physics, Roskilde University, PO Box: 260, 4000 Roskilde, Denmark - e-mail: sjafari@ruc.dk

(received: 22/4/2003 ; accepted: 3/7/2004)

Abstract

In this paper, we present and study the notion of firm $\theta$-continuity to investigate $\theta$-compactness.

**Keywords:** $\theta$-open, $\theta$-compact, quasi $\theta$-continuous, firmly $\theta$-continuous.

1. Introduction

In 1987, Noiri and Popa introduced and investigated a new class of functions called quasi $\theta$-continuous functions. The second author of the present paper (Jafari, 1998) further investigated quasi $\theta$-continuity and also introduced the notion of $\theta$-compactness by using $\theta$-open sets introduced by Velicko (1968). Kupka (1998) inspired by a number of characterizations of $UC$ spaces (also called Atsuji spaces) (Waterhouse, 1965) to characterize compact spaces. For this purpose, he asked the question that what kind of continuity should replace uniform to be sufficiently strong to characterize compact spaces. He was able to tackle this problem by introducing a new type of continuity called firm continuity by which he obtained several characterizations of compact spaces.

It is the aim of this paper to continue the work of Kupka (1998) and obtain some characterizations of $\theta$-compact spaces. In this relation we introduce the notion of firm $\theta$-continuity which is natural for $\theta$-compact spaces.

In what follows we denote the interior and the closure of a subset $A$ of a topological space $(X, \tau)$ by $Int(A)$ and $Cl(A)$, respectively. A point $x \in X$ is called a $\theta$-adherent point of a subset $A$ of $X$ if $A \cap Cl(V) \neq \emptyset$ for every open set $V$ containing $x$. The set of $\theta$-adherent points of $A$
is called the \( \theta \)-closure of \( A \) which is denoted by \( Cl_\theta (A) \). A subset \( A \) of \( X \) is called \( \theta \)-closed if \( A = Cl_\theta (A) \). The complement of a \( \theta \)-closed set is called a \( \theta \)-open set. A topological space \( (X, \tau) \) is said to be \( \theta \)-compact (Jafari, 1998) if every cover of \( X \) by \( \theta \)-open sets has a finite subcover. A subset \( A \) of a topological space \( (X, \tau) \) is said to be \( \theta \)-compact relative to \( \tau \) (Jafari, 1998) if for any cover \( \{ V_i \mid i \in I \} \) of \( A \) by \( \theta \)-open sets of \( (X, \tau) \) there exists a finite subset \( I_0 \) of \( I \) such that \( A \subseteq \bigcup \{ V_i \mid i \in I_0 \} \). A topological space \( (X, \tau) \) is called \( \theta \)-T\(_1\) (Jafari, 1998) if for every pair of distinct points \( x \) and \( y \) in \( X \), there exists a \( \theta \)-open set containing one of the points but not the other. A function is said to be quasi \( \theta \)-continuous (Noiri and Popa, 1987) if every inverse image of each \( \theta \)-open set in the codomain is \( \theta \)-open in the domain.

2. Characterizations of \( \theta \)-compact spaces

**Definition 1:** A function \( f : X \rightarrow Y \), where \( X \) and \( Y \) are topological spaces, is said to have property \( \theta \) if for every \( \theta \)-open cover \( \nabla \) of \( Y \) there exists a finite cover (the members of which need not be necessarily \( \theta \)-open) \( \{ A_1, A_2, \ldots, A_n \} \) of \( X \) such that for each \( i \in \{ 1, 2, \ldots, n \} \), there exists a set \( U_i \in \nabla \) such that \( f(A_i) \subseteq U_i \).

**Lemma 2.1:** A topological space \( X \) is \( \theta \)-compact if and only if for every topological space \( Y \) and every quasi \( \theta \)-continuous function \( f : X \rightarrow Y \), \( f \) has the property \( \theta \).

**Proof:** Let the topological space \( X \) be \( \theta \)-compact and the function \( f : X \rightarrow Y \) be quasi \( \theta \)-continuous. Suppose that \( \Xi \) is a \( \theta \)-open cover of \( Y \). The set \( f(X) \) is \( \theta \)-compact relative to \( Y \). This means that there exists a finite subfamily \( \{ U_1, U_2, \ldots, U_n \} \) of \( \Xi \) which covers \( f(X) \). Then the sets \( A_1 = f^{-1}(U_1), A_2 = f^{-1}(U_2), \ldots, A_n = f^{-1}(U_n) \) form a cover of \( X \) such that \( f(A_i) \subseteq U_i \) for each \( i \in \{ 1, 2, \ldots, n \} \).

Conversely, suppose that \( X \) is a topological space such that for every topological space \( Y \) and every quasi \( \theta \)-continuous function \( f : X \rightarrow Y \), \( f \) has property \( \theta \). It follows that the identity function \( id_X : X \rightarrow X \).
has also property \( \setminus \). Hence, for every \( \theta \)-open cover \( \Xi \) of \( X \), there exists a finite cover \( A_1, A_2, \ldots, A_n \) of \( X \) such that for each \( i \in \{1, 2, \ldots, n \} \) there exists a set \( U_i \in \Xi \) such that \( A_i = id_X(A_i) \subseteq U_i \). Then \( U_1, U_2, \ldots, U_n \) are finite \( \theta \)-subcover of \( \Xi \). Since \( \Xi \) was an arbitrary \( \theta \)-open cover of \( X \), the space \( X \) is \( \theta \)-compact.

Now we introduce the new class of firmly \( \theta \)-continuous functions similar to the class of firmly continuous functions defined in (Kupka, 1998).

**Definition 2:** A function \( f : X \to Y \) is called *firmly \( \theta \)-continuous* if for every \( \theta \)-open cover \( \nabla \) of \( Y \) there exists a finite \( \theta \)-open cover \( \Xi \) of \( X \) such that for every \( U \in \Xi \), there exists a set \( G \in \nabla \) such that \( f(U) \subseteq G \).

**Remark 2.2:** It should be noted that if the topological space \( X \) is \( \theta \)-compact and \( Y \) is an arbitrary topological space, then every quasi \( \theta \)-continuous function \( f : X \to Y \) is firmly \( \theta \)-continuous.

**Lemma 2.3:** Let \( X, Y, Z \) and \( W \) be topological spaces. Let \( g : X \to Y \) and \( h : Z \to W \) be quasi \( \theta \)-continuous functions and let \( f : Y \to Z \) be firmly \( \theta \)-continuous. Then the functions \( f \circ g : X \to Z \) and \( h \circ f : Y \to W \) are firmly \( \theta \)-continuous.

**Lemma 2.4:** Let \( X \) and \( Y \) be topological spaces. Suppose that \( f : X \to Y \) is a quasi \( \theta \)-continuous function which has the property \( \setminus \). Then \( f \) is firmly \( \theta \)-continuous.

**Theorem 2.5:** The following statements are equivalent for a topological space \( (X, \tau) \):

1. \( X \) is \( \theta \)-compact;
2. The identity function \( id_X : X \to X \) is firmly \( \theta \)-continuous;
3. Every quasi \( \theta \)-continuous function from \( X \) to \( X \) is firmly \( \theta \)-continuous;
4. Every quasi \( \theta \)-continuous function from \( X \) to a topological space \( Y \) is firmly \( \theta \)-continuous;
5. Every quasi \( \theta \)-continuous function from \( X \) to a topological space \( Y \) has the property \( \setminus \);
6. For each topological space \( Y \) and each quasi \( \theta \)-continuous function \( f : Y \to X \), \( f \) is firmly \( \theta \)-continuous.
Proof. (1) ⇒ (2) : Let \( X \) be \( \theta \)-compact. The identity function \( id_X: X \to X \) is quasi \( \theta \)-continuous and by Remark 2.2 \( id_X \) is firmly \( \theta \)-continuous.

(2) ⇒ (3) : Let \( f: X \to X \) be any quasi \( \theta \)-continuous function. By (2), the identity function \( id_X : X \to X \) is firmly \( \theta \)-continuous.

Therefore by Lemma 2.3 \( f = id_X \circ f: X \to X \) is firmly \( \theta \)-continuous.

(3) ⇒ (4) : Suppose that \( f: X \to Y \) is any quasi \( \theta \)-continuous function. The identity \( id_X: X \to X \) is quasi \( \theta \)-continuous and by (3) \( id_X \) is firmly \( \theta \)-continuous. As a consequence of Lemma 2.3, we have that \( f = id_X \circ f: X \to Y \) is firmly \( \theta \)-continuous.

(4) ⇒ (5) : Obvious.

(5) ⇒ (1) : This is an immediate consequence of Lemma 2.1.

(6) ⇒ (2) : Suppose that \( id_X: X \to X \) is the identity function. Then \( id_X \) is quasi \( \theta \)-continuous and by (6) \( id_X \) is firmly \( \theta \)-continuous.

(1) ⇒ (6) : Suppose that \( \nabla \) is a \( \theta \)-open cover of \( X \). Since \( X \) is \( \theta \)-compact, then there is a finite \( \theta \)-subcover \( U_1, U_2, \ldots, U_n \) of \( \nabla \).

Let \( A_i = f^{-1}(U_i) \) for \( i \in I \), where \( I = \{ 1, 2, \ldots, n \} \). We have that \( f(A_i) \subseteq U_i \) for every \( i \in I \). Therefore \( f \) is firmly \( \theta \)-continuous.

**Theorem 2.6:** If \( f: X \to Y \) is a firmly \( \theta \)-continuous function, where \( X \) is a topological space and \( Y \) is a \( \theta \)-\( T_1 \) topological space, then \( f \) is quasi \( \theta \)-continuous.

Proof. Let \( x \) be an arbitrary point of \( X \) and \( V \) be a \( \theta \)-open set of \( Y \) containing \( f(x) \). We define a \( \theta \)-open cover \( \Xi \) of \( Y \) such that \( \Xi = \{ V, Y - f(x) \} \). Since \( f \) is firmly \( \theta \)-continuous, it follows that there exists a finite \( \theta \)-open cover \( \{ P_1, P_2, \ldots, P_n \} \) of \( X \) such that \( f(P_j) \subseteq V \) or \( f(P_j) \subseteq Y - f(x) \) for every \( i \in \{ 1, 2, \ldots, n \} \). Let \( x \in P_j \) for
some index \( j \). Since \( f(P_j) \) contains \( f(x) \), so it follows that \( f(P_j) \subseteq V \).

This shows that \( f \) is quasi-\( \theta \)-continuous.

**References**


