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On a Non-Uniformly Elliptic System of PDE and Boundary Value Problems

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Abstract

The complex form of two first order real equations with a linear boundary condition, existence and uniqueness of the solution of the boundary value problem is studied using a function theoretic method.

Keywords: *Complex functions, System of partial differential equations in the plane, Non-Uniformly elliptic Lavrentiev type, Boundary values problem*

I. Introduction

A nonlinear system of Lavrentiev type equations with degeneration of ellipticity together with a linear boundary condition is investigated. By application of a function theoretic method in partial differential equations, existence, uniqueness and stability of the solution in the Sobolev space $W_{1,p}$ are introduced.

1. Preliminaries

We shall confine ourselves to the boundary value problem for the nonlinear system type of two first order real equations

$$\varphi_i (x, y, u, v, u_x, u_y, v_x, v_y) = 0 \qquad i = 1,2 \qquad (1.1)$$

with degeneration of ellipticity. As it is well-known the methods of complex function theory have a wide use in many questions of mathematical analysis. The possibility and importance of employing complex variable methods in PDE is so wide that it presents a real difficulty to give a survey of them. For a great many references one may consult for instance the books of Dzhuraev (1999) and Lanckau & Tutschke (1985) or Wen & Begehr (1990). System (1.1) with two unknown u(x,y) and v(x,y) of two independent variables x,y in the plane, can be written in the complex form

$$w_{\bar{z}} = H\left(z, w, w_z\right) \tag{1.2}$$

(Dzhuraev, 1999), where z = x + iy, w = w(z) = u(x, y) + iv(x, y) and $w_{\bar{z}} = \dfrac{\partial w}{\partial \bar{z}} = \left(\dfrac{\partial w}{\partial x} + i\dfrac{\partial w}{\partial y}\right)\Big/2, w_z = \dfrac{\partial w}{\partial z} = \left(\dfrac{\partial w}{\partial x} - i\dfrac{\partial w}{\partial y}\right)\Big/2$. Obviously, equation (1.2) contains the complex form of the Cauchy-Riemann system $w_{\bar{z}} = 0$ and the well-known Beltrami's equation $w_{\bar{z}} = q(z)w_z$.

Let *L* be the boundary contours of a Liapounoff region *G* and ∧ be another system of finite non-intersecting contour inside *G* which decomposes *G* into a finite number of regions; the union of all these subsets of the domain *G* will be called the domain *D*. *L* and ∧ have no common points. Let us consider the equation (1.2) in the domain *D*, satisfying the boundary conditions

$$\operatorname{Re}\left[\overline{\lambda(t)}\ w(x)\right] = \gamma(t) \tag{1.3}$$

on *L*, and

$$w^+(t) = a(t)w^-(t) + b(t) \tag{1.4}$$

on ∧. The symbols w^+ and w^- are understood in the usual sense of the theory of Hilbert boundary values problem, λ, γ and *a, b* are given functions on *L* and ∧ respectively. The function $H(z,w,w_z)$ satisfies the Lipschitz condition

$$\left|H(z,w,\xi_1) - H(z,w,\xi_2)\right| \le q(z,w)\ \left|\xi_1 - \xi_2\right| \tag{1.5}$$

$$0 \le q(z,w) \le q_0 < 1.$$

Then, equation (1.2) satisfying (1.4) is called uniformly elliptic in the sense of Lavren-tiev in the domain considered (Mamourian, 1997).

Let us recall that with some natural assumptions on the coefficients:

Holder continuity of λ, γ and *a, b* on the boundaries *L* and ∧, respectively; *L* is the boundary of Liapounoff region *G*; ∧ belongs to the class C^1, and the solution being sought in the class of sectional continuous functions in *D*, which have continuous extensions up to the boundary and belonging to the class $W_{1,p}(D), p > 2$. It has been proved that, boundary value problem (1.2), (1.3), (1.4) for (1.2) in *D*, can be reduced to a boundary values problem of the type (1.2), (1.3) in *G*. To avoid a long expression, we shall not bring here the proof (see for instance Lanckau & Tutschke (1989) or Mamourian (1975).

Remark 1.1. In general, the exponent ν of the Hölder continuity of *a, b* on ∧ will be assumed to be $\frac{1}{2} < \nu < 1$, but in the case when $H = 0(w_{\bar{z}} = 0)$, or a larger class of functions, i.e. Generalized analytic functions, we can assume 0 < ν < 1 (Mamourian, 1975).

2. Non-uniformly ellipticity of the equation

In this part, an extension to Lavrentiev's condition is introduced (see also Mamourian, 1997), Hence instead of *q* in (1.5) which assumed to be a real function of complex variables z, w, suppose that *q* be a real function of complex variables *z, ξ, η*. We confine ourselves to the nonlinear system of equations

$$W_{\bar{z}} = \widetilde{H}(z, w_z) = \Phi(\theta(z) w_z) + F(z) \,, \tag{2.1}$$

in *G*, where the function $\widetilde{H}(z, w_z)$ satisfies the following inequality

$$\left|\widetilde{H}(z,\xi) - \widetilde{H}(z,\eta)\right| \le q(z,\xi,\eta)|\xi - \eta| \tag{2.2}$$

$$q(z,\xi,\eta) \le \widetilde{q}(|\xi - \eta|) \le 1 .$$

Hypothesis A: In (2.2), the function $\widetilde{q}$ as a function $\mu\,(\mu = |\xi - \eta|)$ is continuous in [0, ∞); $\widetilde{q}(\mu) < 1$ for $\mu \in (0, \infty)$; the function $\mu.\widetilde{q}^{\,2}(\mu)$ is increasing and concave; the complex function $\theta(z)$ is assumed to be measurable belonging to $L_\infty(G)$.

The index corresponding to boundary values problem (2.1), (1.3) is defined by:

$$n = ind\lambda = \frac{1}{2\pi i}\int_L d\left(\log \lambda(t)\right) \tag{2.3}$$

Let us recall that, in the classical Riemann-Hilbert boundary value problem of the type (2.1) ,(1.3), relative to the uniformly ellipticity of equations of Lavrentiev type, the solution w is sought in the Sobolev space $W_{1,P}(G)$, for some p > 2. In the case of equation (2.1) with degeneration of ellipticity (2.2), the L_p- theory has not been used directly for the proof of the existence of the solution of problem (2.1), (1.3). Therefore formulation of the boundary values problem, involves the weak boundary condition (Mamourian, 1997).

We shall consider the equation (2.1) fulfilling the boundary condition (1.3) on L with usual assumptions on the coefficients; the complex function λ and real function γ are Hölder continuous on boundary *L* with respect to τ, where $0 < \tau \leq 1$.*G* be a multiply connected domain of Liapounoff type, $|\theta(z)| \leq 1$; the complex function *F*(z) assumed to be measurable belonging to the class $L_p(G)$, for some p > 2. The solution w will be sought in the Sobolev space $W_{1,P}(G), p >$ 2.

It is well-known that if Φ = 0, and n ≤ 0, the non-homogeneous boundary values problem (2.1), (1.3) is solvable, if and only if

$$\frac{1}{2i}\int_L \lambda(t)\psi(t)\gamma(t)dt = \mathrm{Re}\left[\int_G \psi(z)F(z)d\sigma_z\right] \tag{2.4}$$

where ψ is an arbitrary solution of the homogeneous problem adjoint to problem, (2,l), (1,3) (Lanckau and Tutschke, 1989, pp. 98-101).

Proposition 2.1 Under hypothesis A, if n < 0, problem (2.1), (1.3) has a solution and it is unique.

Making use of representation formula for the solution w of the problem (2.1), (1.3):

$$w = T(\rho)_{(z)} + \varphi(z) \tag{2.5}$$

(Mamourian, 1997), where $T(\rho) = (T(\rho))_G$, and $\varphi(z)$ is the solution of the boundary value problem (2.1),(1.3) in the case when $\Phi = 0$, we observe that T, depending oh the index n, fulfils the homogeneous boundary condition corresponding to (1.3) on L, when $z \to t(z \in D, t \in L)$.

Remark 2.1. If $m = 0$, for explicit form of T see for instance Begehr and Hsiao (1983). Moreover $\partial T(\rho)/\partial \bar{z} = \rho(z)$. Denoting by: $S(\rho) = \partial T(\rho)/\partial z$. Since $n < 0$, we conclude that the L_2-norm of S is equal to one, also S is a bounded operator from $L_p(G)$, $p > 1$ into itself and the continuity of $\|S\|_{L_p(G)}$ with respect tp $p \geq 1$ can be proved through the well-known Riesz-Thorin convexity theorem.

Remark 2.2. The norm $\| \quad \|$ is determined by

$$\|\rho\|_{L_2(G)} = \left(\frac{1}{|G|} \int_G |\rho(z)|^2 \, d\sigma_z \right)^{\frac{1}{2}}$$

In view of (2.1),(1.3),(2.5), we obtain the following equation for ρ:

$$\rho = \Phi(\theta S(\rho) + \theta\varphi'). \tag{2.6}$$

Let us assume that $\rho \in L_2(G)$. Then integral equation (2.6) can be solved through a successive approximation method. According to hypothesis A, we obtain the inequality

$$\|\rho_1 - \rho_2\| \leq \tilde{q}(\|\rho_1 - \rho_2\|) \, \|\rho_1 - \rho_2\| \tag{2.7}$$

and the existence, also uniqueness of the solution of integral equation in $L_2(G)$ can be proved. We shall prove that the solution $w \in W_{1,2}(G)$ of the problem (2.1),(1.3) actually belongs to $W_{1,p}(G)$ for some $p > 2$. The number

$$q_0 = \lim_{\rho \to \infty} \sup(\tilde{q}(\rho)) < 1 \tag{2.8}$$

is called the coefficient of ellipticity corresponding to boundary value problem (2.1),(1.3). q_0 shows that how fast the gradient may approach infinity and consequently q_0 will influence the exponent $p > 2$ of the integrability of the gradient.
Let p be such that

$$q_0 \|S\|_{L_p(G)} < 1 . \tag{2.9}$$

Proposition 2.2 Under hypothesis A,(2.4) and inequality (2.9), the solution w of problem (2.1), (1.3) belongs to the Sobolev space $W_{1,p}(G)$, $p > 2$.

In view of the integral equation (2.6), if we write

$$\rho_{j+1} = \Phi(\theta S(\rho_j) + \theta\varphi') \tag{2.10}$$

$\rho_0 = 0$, $j = 0,1, \ldots$, since ρ_j converges to ρ in L_2, it is sufficient to show the uniform estimates for $\|\rho_j\|_{L_P}(G)$. According to the relation (2,10), $p_j(j = 0, 1, \ldots)$ is in $L_p(G)$. Clearly $\rho_0 \in L_p(G)$, then by inequality (2.2), and properties of $\tilde{q}$, we have

$$|\rho_{j+1}| \le |\theta S(\rho_j)| + |\theta\varphi'| \in L_p(G), \tag{2.11}$$

which is derived by induction and L_p continuity of S. moreover because of continuity of $\|S\|_{L_p(D)}$ relative to p, a number $\alpha \in (q_0, 1)$ exists such that

$$\alpha . \|S\|_{L_p(D)} < 1 \tag{2.12}$$

In view of (2.8),(2.11), some calculation and integration leads to the following inequality

$$\|\rho_{j+1}\|_{L_p(G)} \le \alpha . \|S\|_{L_p(G)} . \|\rho_j\|_{L_p(G)} + M(mesG)^{\frac{1}{\rho}} + \|\varphi'\|_{L_p(G)} \tag{2.13}$$

where M is a constant, and we obtain

$$\left\|\rho_{j+1}\right\|_{L_p(G)} \leq \frac{1}{1-\alpha.\left\|S\right\|_{L_p(G)}}\left[M\left(mesG\right)^{\frac{1}{p}}+\left\|\varphi'\right\|_{L_p(G)}\right] \quad (2.14)$$

We observe that the right-hand side of inequality (2.14) does not depend on j, which indicates the uniform estimate of $||\rho_{j+1}||_{L_p(G)}$. Therefore we conclude that $\rho \in L_P(G)$ and the upper bound of $||\rho||_{L_p(D)}$ will be the right hand side of inequality(2.14). Since $\varphi \in W_{1,p}(G)$ and T maps $L_p(G)$ into $W_{1,p}(G)$, the solution w = $T(p)$ + φ belongs to $W_{1,p}(G)$.

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