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A New Method for Solving Time Optimal Control Problems

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Abstract

In this paper we consider the *Time Optimal Control Problem with Bounded state* (TOCPB). By means of a process of embedding and using measure theory, this problem is replaced by another, in which we seek to minimize a linear form over a subset of a measure space defined by linear equalities. The theory allows us to convert the new problem to an infinite-dimensional linear programming problem. Afterwards, the infinite-dimensional linear programming problem is approximated by a finite dimensional one. Then by the solution of the final linear programming problem one can find an approximate value of the trajectory function $x(\cdot)$, control function $u(\cdot)$ and optimal time T as well.

AMS classification(49A).

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1. Introduction

The control problems with bounded state (**CPB**) and optimal control problems with bounded state (**OCPB**) have received considerable attention from theoretical and numerical points of view (Dixon & Biggs, 1972; Maurer & Gillessen, 1975; Dixon & Biggs, 1981; Vlassenbroeck, 1988; Fraser-Andrews, 1996). We used measure theory to solve **OCPB**'s (Heydari, *et al.,* 2001). In the last decade some authors used this method to solve optimal control problems in lumped and distributed systems (Heydari, *et al.,* 2001; Rubio, 1986; Kamyad *et al.,* 1992; Farahi, *et al., 1996a;* Farahi, *et al., 1996b;* Heydari, *et al.,* 1999; Alavi *et al*., 1997;Effati & Kamyad, 1998).

In this article we consider the following **TOCPB**:

$$\textit{Minimize} \quad T \tag{1}$$

subject to: $\dot{x} = g(x,u)$, $\quad (x,u) \in A \times U$

$$\eta(x(t)) \le 0 \ , \qquad t \in J \tag{2}$$

$$x(0) = x_0 \ , \quad x(T) = x_1.$$

Where x_0 and x_1 are given and $J = [0,T]$, **A** is a bounded, closed, and piecewise connected set in R^n such that $x(t) \in A, \forall t \in J$ and U is a bounded and closed set in R^n where $u(t) \in U, \forall t \in J$. The scalar function η is assumed to be $(p+1)$–times continuously differentiable in x, and the function g is assumed to be $(p+1)$– times continuously differentiable in x and u where p is definite.

We may rewrite the objective function (1) as follows:

$$\text{Minimize } I(.) := \int_J dt. \tag{3}$$

The region specified by

$$\eta(x(t)) = 0 \ , \tag{4}$$

is called the state boundary. Differentiating (4) and using (2) gives, $\eta^{(i)}$, the ith times derivative of η. We say the problem is of order of p if :

$$\eta^{(i)}(x(t)) = 0 \ , \ i = 0,\ 1,\ \ldots,\ p-1, \quad \eta^{(p)}(x(t),u(t)) = 0. \tag{5}$$

Since also in this problem we have $\eta \le 0$, thus

$$s\,\eta(x(t)) = \eta(x(t)) + |\eta(x(t))| = 0. \tag{6}$$

Now define the pair $[x(\cdot),u(\cdot)]$ to be an *admissible pair*, if:

1) The function $x(\cdot)$ is continuous, and $x(t) \in A, \forall t \in J$.
2) The function $u(\cdot)$ is Lebesgue measurable, and $u(t) \in U, \forall t \in J$.
3) The pair $[x(\cdot),u(\cdot)]$ satisfies differential equation (2) and relations (5) and (6) a.e. on $J^\circ = (0,T)$ in the sense of Cara'theodory.

We denote the set of admissible pairs by W. The problem has no solution unless W≠0. By this assumption now the problem is as follows:
Find an optimal admissible $w \in W$ which minimizes the functional:

$$I(\cdot) = \int_J dt .$$

2. Metamorphosis

Assume that B is an open ball in R^n containing A, denote the space of all differentiable function on B by $C^{'}(B)$, and define:

$$\phi^g = \nabla \phi(x).g \tag{7}$$

where $\nabla\phi(.)$ and $g(.,.)$ are n-vectors and the right-hand side of (7) presents an inner product, ϕ^g is in the space $C(\Omega)$ of real-valued continuous functions defined on the compact set $\Omega = J \times A \times U$. Then by the definitions of g and ϕ and using the chain rule we have:

$$\int_J \phi^g(x(t),u(t))dt = \int_J \dot{\phi}(x(t))dt$$

$$= \phi(x(T)) - \phi(x(0)) \equiv \delta\phi, \forall \phi \in C^{'}(B). \tag{8}$$

Since A may have an empty interior in R^n , we need to introduce the set B and space $C^{'}(B)$. Let $D(J^\circ)$ be the space of infinitely differentiable real valued functions with compact support in J°, and each x and g have n components such as x_j and $g_j, j = 1,2,\ldots,n$.

For each $\psi \in D(J^\circ)$ define:

$$\psi^j(t,x(t),u(t)) = x_j\dot{\psi}(t) + g_j\psi(t), j = 1,2,\ldots,n. \tag{9}$$

If w is an admissible pair, then for any $\psi \in D(J^\circ)$ we have:

$$\int_J \psi^j(t,x(t),u(t))dt = \int_J x_j\dot{\psi}(t)dt + \int_J g_j\psi(t)dt$$

$$= x_j(t)\psi(t)\,|_0^T - \int_J \{\dot{x}_j - g_j(t,x(t),u(t))\}\psi(t)dt.$$

Since ψ has compact support on J°, so

$$\psi(0) = \psi(T) = 0,$$

and since $\dot{x} = g$, so

$$\int_J \psi^j(t,x(t),u(t))dt = 0. \tag{10}$$

Now assuming that F is an open ball in R containing J, denote the space of all differentiable functions on F by $C^1(F)$. Set

$$\beta^g(t,x,u) = \dot{\beta}(t),(t,x,u) \in \Omega\,, \tag{11}$$

and

$$\int_J \beta(t,x,u)dt = \alpha_\beta, \beta \in C^1(F)\,, \tag{12}$$

where a_β is the Lebesgue integral of $\beta(t,x,u)$ on J. Also by (5),

$$\int_J \eta^{(i)}(x(t))dt = 0\,,\ i = 0,1,\ldots,p-1 \tag{13}$$

and finally by (6),

$$\int_J s\eta(x(t))dt = 0. \tag{14}$$

Now consider the mapping:

$$\Lambda_w : F(.,.,.) \in C(\Omega) \to \int_J F(t,x(t),u(t))dt \tag{15}$$

that is a linear positive functional. The left-hand sides of the equalities (8)-(10)-(12)-(13)-(14) are all integrals, thus by using these equalities and (15), the problem (2)-(3) can be modified as:

$$\text{Minimize}\quad \Lambda_w(1) \tag{16}$$

subject to:

$$\begin{aligned}
&\Lambda_w(\phi^g) = \delta\phi, \phi \in C^{'}(B)\\
&\Lambda_w(\psi^j) = 0,\, j = 1,2,\ldots,n; \psi \in D(J^\circ)\\
&\Lambda_w(\beta) = \alpha_\beta, \beta \in C^1(F).\\
&\Lambda_w(\eta^{(i)}) = 0, i = 0,1,\ldots,p-1,\\
&\Lambda_w(s\eta) = 0.
\end{aligned} \tag{17}$$

We mention that Λ_w is appositive Radon measure on the set $C(\Omega)$. Denote the space of all positive Radon measure on Ω by $R^+(\Omega)$. A Radon measure on Ω can be identified with a regular Borel measure on this set (see Royden, 1995, Riesz representation Theorem). Thus, for a given positive functional Λ_w on $C(\Omega)$, there is a positive Borel measure on Ω such that:

$$\Lambda_w(F) = \int_\Omega F d\mu = \mu(F), F \in C(\Omega).$$

Now, the problem (16)-(17) can be replaced by the following new problem. We seek a measure in $M^+(\Omega)$ (the space of all positive Borel measures on Ω)which minimizes functional

$$\mu \in M^+(\Omega) \to \mu(1) \in R \tag{19}$$

and satisfies the following constraints:

$$\begin{aligned} &\mu(\phi^g) = \delta\phi\,,\ \ \phi \in C'(B) \\ &\mu(\psi^j) = 0, j = 1,2,\ldots,n; \psi \in D(J^\circ) \\ &\mu(\beta) = a_\beta, \beta \in C^1(F). \\ &\mu(\eta^{(i)}) = 0, i = 0,1,\ldots,p-1, \\ &\mu(s\eta) = 0. \end{aligned} \tag{19}$$

Now consider the extension of our problem: we shall consider the minimization of (18) over the set Q of all positive Borel measures on Ω satisfying (19). The main advantages of considering this measure theoretic from of the problem is: "*The existence of an optimal measure in the set Q that satisfies (18)-(19) can be studied in a straightforward manner without having to impose conditions such as convexity which may be artificial*."

By the proposition II.1, Theorem II.1 and proposition II.3 of (Rubio, 1986), we can prove the existence of the optimal measure in the new set Q.

3. First approximation

The problem (18)-(19) is an infinite dimensional linear programming (LP) problem, because all of the functionals in (18)-(19) are linear in the variable μ, and furthermore, the measure μ is required to be positive. We note that this is true even if the original problem is nonlinear, linearity in the present sense was gained by the consideration of admissible pairs as positive measures on Ω. Of course, it is an infinite dimensional LP problem, because $M^+(\Omega)$ is infinite dimensional space. It is possible to approximate the solution of this problem by the solution of a finite- dimensional LP of sufficiently large dimension. Also, from the solution of this new finite dimensional LP we induce an approximated admissible pair in a suitable manner. We shall first develop an intermediate problem, still infinite-dimensional by considering the minimization (18); not over the set Q but over a subset of $M^+(\Omega)$ with only a finite numbers of the constraints in (19) being satisfied. This will be achieved by choosing countable sets of functions whose linear combinations are dense in the set Q, and then selecting a finite number of them.

Consider the first set of equalities in (19). Let the set $\{\phi_i, i = 1,2,\ldots\}$ be such that the linear combinations of the functions $\phi_i \in C'(B)$ are uniformly dense in $C'(B)$. For instance, these functions can be taken to be monomials in the components of the n-vectors x.

If in the second stage, ψ_r's are chosen as below:

$$\sin[2\pi r t/\delta t],\;\; 1-\cos[2\pi r t/\delta t],\;\; r = 1,2,\ldots \tag{20}$$

where $\delta t = T$, as in the next section, then the problem is converted to a non-linear programming (NLP). Since the solution of the NLP problem is difficult by our disposal, we would define ψ_r's such that the problem be an LP problem. By using controllability we consider t_a such (as we assumed) that $0 < t_a < T$ and define:

$$\psi_r(t) = \begin{cases} \sin[2\pi r t/\delta t] & t < t_a \\ 0 & otherwise \end{cases}$$

or

$$\psi_r(t) = \begin{cases} 1-\cos[2\pi r t/\delta t] & t < t_a \\ 0 & otherwise \end{cases} \tag{21}$$

where $\delta t = t_a$ and $r = 1,2,\ldots$.

4. Secound approximation

The first approximation will be completed by using above subjects and the following proposition.

We mention that $\beta(t)$'s and $\psi(t)$'s are special cases of function ϕ that depend only t, then:

Proposition 1: Consider the linear program consisting of the minimizing function $\mu \to \mu(1)$ over the set Q_M of measures in $M^+(\Omega)$ satisfying:

$$\mu\left(\phi_b^g\right) = \delta\phi, b = 1,2,\ldots,M$$
$$\mu\left(\eta^{(i)}\right) = \circ\ , \quad i = \circ,\ 1,\ \ldots,\ p-1,$$
$$\mu\left(s\eta\right) = \circ\ .$$

then $\lambda_M \equiv \inf_{QM} \mu(1)$ tends to $\lambda \equiv \inf_Q \mu(1)$ as $M \to \infty$.

Proof:

(a) Exists ξ such that λ_M tends to ξ as $M \to \infty$:

We know that:

$$Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_M \supseteq \cdots \supseteq Q,$$
$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_M \le \cdots \lambda,$$

$(\lambda_M)_{M=1}^{\infty}$ is non decresing and bounded sequence then converges to a number ξ such that $\xi \le \lambda$.

(b) $\xi = \lambda$:

set $R \equiv \bigcap_{M=1}^{\infty} Q_M$ then $R \supseteq Q$ and $\xi = \inf_R \mu(1)$. We show that $R \subseteq Q$:

if $\mu \in R$, then:

$$\mu(\phi^g) = \delta\phi, \forall\, \phi \in C'(B):$$

We have $\mu(\phi^g) = \delta\phi, \forall\, \phi \in span\{\phi_b, b = 1,2,\ldots\}$, since there is the sequence $(\phi_k) \in C'(B)$ such that S_1, S_2 and S_3 tend to zero as $k \to \infty$ where:

$$S_1 = \sup\left|\phi_t(t,x) - \phi_{k_t}(t,x)\right|$$
$$S_2 = \sup\left|\phi_x(t,x) - \phi_{k_x}(t,x)\right|$$
$$S_3 = \sup\left|\phi(t,x) - \phi_k(t,x)\right|$$

then

$$\left|\mu(\phi^g)-\delta\phi\right|=\left|\mu(\phi^g)-\delta\phi-\mu(\phi_k^g)+\delta\phi_k\right|=$$
$$\left|\int\{\left|\phi_x(t,x)-\phi_{k_x}(t,x)\right|g(t,x,u)+[\phi_t(t,x)-\phi_{k_t}(t,x)]\}d\mu-(\delta\phi-\delta\phi_k)\right|\leq$$
$$\alpha_1\sup\left|\phi_t(t,x)-\phi_{k_t}(t,x)\right|+\alpha_2\sup\left|\phi_x(t,x)-\phi_{k_x}(t,x)\right|+\sup\left|\phi(t,x)-\phi_k(t,x)\right|=$$
$$\alpha_1 S_1+\alpha_2 S_2+S_3,$$

where $\Omega=J\times A\times U,\alpha_1=\int_\Omega g(t,x,u)\,d\mu$ and $\alpha_2=\int_\Omega d\mu$.

Since the right-hand side of the above inequality tends to zero as $k\to\infty$, while left-hand side is independent of *k*, therefore:

$$\mu(\phi^g)=\delta\phi,\forall\phi\in C'(B)$$

Thus if $\mu\in R$, then it is also in *Q*, from which $\xi\leq\lambda$, and the contention of the proposition follows. □

Proposition 2: The measure μ^* in the set Q_M at which the function $\mu\to\mu(1)$ attains its minimum has the form:

$$\mu^*=\sum_{k=1}^{M}\alpha_k^*\delta\left(y_k^*\right) \tag{22}$$

$y_k^*\in\Omega$ and $\alpha_k^*\geq 0,k=1,2,\ldots,M,$ and $\delta(.)$ is unitary atomic measure with the support being the singleton set $\left\{y_k^*\right\}$, characterized by:

$$\delta(y)(F)=F(y),y\in\Omega..$$

Proof: similar to Proposition III.3 in (Heydari, *et al.,* 2001).
This structural result points the way towards a nonlinear problem in which the unknowns are the coefficients α_k^* and supports $\left\{y_k^*\right\},k=1,2,\ldots,M$.

To change this problem to an LP, we use another approximation. If ω^N is a countable dense subset of Ω, we can approximate μ^* by a measure $\nu\in M^+(\Omega)$ such that:

$$\nu=\sum_{k=1}^{M}\alpha_k^*\delta\left(y_k\right)$$

Where $y_k\in\omega^N=\left\{y_1,y_2,\ldots,y_N\right\}$ [Proposition III.3 of (Rubio, 1986)].

This result suggests the following LP problem:
Given $\varepsilon > 0$ and $y_k \in w^N, k = 1,2,\dots,N$,

$$Minimize \qquad \sum_{k=1}^{N} \alpha_k \tag{23}$$

Subject to:

$$\begin{aligned}
&\left|\sum_{k=1}^{N} \alpha_k \phi_b^g(y_k) - \delta\phi_b\right| \le \varepsilon, \quad b = 1,\ 2,\ \cdots,M_1,\\
&\left|\sum_{k=1}^{N} \alpha_k \psi_r^j(y_k)\right| \le \varepsilon, \quad \begin{matrix} j = 1,2,\dots,n \\ r = 1,2,\dots,M_2/n \end{matrix}\\
&\left|\sum_{k=1}^{N} \alpha_k \beta_s(y_k) - \alpha\beta_s\right| \le \varepsilon, \quad s = 1,\ 2,\ \cdots,L,\\
&\left|\sum_{k=1}^{N} \alpha_k \eta^{(i)}(y_k)\right| \le \varepsilon, \quad i = 1,\ 2,\ \cdots,p-1\\
&\left|\sum_{k=1}^{N} \alpha_k s\eta(y_k)\right| \le \varepsilon,\\
&\alpha_k \ge 0,\ k = 1,\ 2,\ \cdots,N.
\end{aligned} \tag{24}$$

Assume $P(M)^\varepsilon$ in R^N shows the set $(\alpha_1,\alpha_2,\dots,\alpha_N)$ where $\alpha_k \ge 0, \quad k = 1,\ 2,\ \cdots,N$ satisfies (24), then by Theorem III.1 of (Rubio, 1986), for every $\varepsilon > 0$ the problem of minimizing the functional (23) on the set $P(M)^\varepsilon$ has a solution for $N = N(\varepsilon)$ sufficiently large, and the solution satisfies

$$\lambda_M + \rho(\varepsilon) \le \sum_{j=1}^{N} \alpha_j \le \lambda_M + \varepsilon$$

where $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Let $\theta_r \in C^1(F)$,

$$\theta_r(t,x,u) = t^r, r = 0,1,\dots \tag{25}$$

then the set of θ_r's is dense in $C^1(F)$. Assume that there are L of them in the set $\{\phi_i^g\}_{i=1}^{M_1}$. It is necessary to choose L number of functions of the time only, to replace the functions $\theta_r, r = 0,1,\dots$ which were not found suitable, so we have chosen some suitable functions, to be denoted by $f_s, s=1,2,\ldots,L$, as follows:

$$f_s(t)=\begin{cases}1 & t\in J_s\\ 0 & otherwise\end{cases}$$

where $J_s=((s-1)d,sd), d=t_a/(L-1), s=1,2,\ldots,L-1$, and $J_L=(t_a,T)$. Since every continuous function can be written as a linear combination of monomials of the type $1,x,x^2,\ldots$, we assume,

$$\phi_1=x_1,\phi_2=x_2,\ldots,\phi_n=x_n,$$
$$\phi_{n+1}=x_1^2,\phi_{n+2}=x_2^2,\ldots,\phi_{2n}=x_n^2$$

until M_1 functions are chosen. Also assume for $r=1,2,\ldots,M_{2_1}$,

$$\psi_r(t)=\begin{cases}\sin[2\pi rt/t_a] & t<t_\alpha\\ 0 & otherwise\end{cases}$$

and for $r=M_{2_1}+1,M_{2_1}+2,\ldots,2M_{2_1}$,

$$\psi_r(t)=\begin{cases}1-\cos[2\pi rt/t_a] & t<t_a\\ 0 & otherwise\end{cases}$$

then we have $M_2=2nM_{2_1}$.

Now, if in the problem (23)-(24), $\varepsilon\to 0,\quad N=N_t\times N_x\times N_u$, where N_t and N_x and N_u are the umbers of partitions on the *t, x* and *u* axes respectively, and $y_j=(t_j,x_j,u_j)\in\omega^N, j=1,2,\ldots,N$, then we have the following LP problem:

$$Minimize \qquad \sum_{k=1}^{N}\alpha_k \tag{26}$$

Subject to:

$$\sum_{k=1}^{N} \alpha_k \phi_b^g(y_k) = \delta\phi_b, \quad b = 1, 2, \cdots, M_1,$$
$$\sum_{k=1}^{N} \alpha_k \psi_r^j(y_k) = 0, \quad \begin{array}{c} j = 1,2,\ldots,n \\ r = 1,2,\ldots,M_2/n \end{array}$$
$$\sum_{k=1}^{N} \alpha_k f_s(y_k) = a_s, \quad s = 1, 2, \cdots, L, \tag{27}$$
$$\sum_{k=1}^{N} \alpha_k \eta^{(i)}(y_k) = 0, \quad i = 1, 2, \cdots, p-1$$
$$\sum_{k=1}^{N} \alpha_k s\eta(y_k) = 0,$$
$$\alpha_k \geq 0, \quad k = 1, 2, \cdots, N \ ,$$

where a_s is the integral of f_s on *J*. By the solution of this finite dimensional LP problem we obtain the nearly optimal α^*'s.

The procedure to construct a piecewise construct control function approximating the action of the optimal measure is based on the analysis in Rubio (Rubio, 1986).

5. Numerical examples

We have estimated the solution of some time optimal control problems with bounded state by using the techniques developed here. Before presenting the result, it necessary to make several comments:

(i) The intervals *A* and *U* are chosen appropriate such that all the problems are controllable.

(ii) The Time interval *J* is chosen as $J = [0, T]$ in all cases.

(iii) The sets of the form $\omega^N = \{y_k, k = 1,2,\ldots,N\}$ were constructed by dividing the appropriate intervals into a number of subintervals, defining in this way a grid of points. We found it is necessary to change these grids according to each problem.

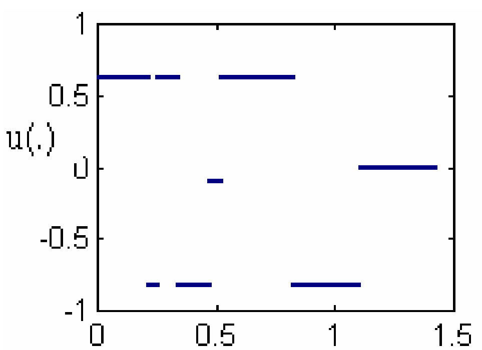
(iv) The number of constraints M_1, M_2 and *L*, are chosen sufficiently large.

(v) The solution of the linear program (26)–(27) were estimated by means of a home – made *revised simplex method*.

Example 1. Minimize $J = T$
subject to

$$\dot{x}_1 = x_2, \qquad x_1(0) = 0, \qquad x_1(T) = 0.5,$$
$$\dot{x}_2 = u, \qquad x_2(0) = x_2(T) == 0$$

We divided the sets $J = [0, T]$, $A_1 = [-1, 0.5]$, $A_2 = [-1,\ 0.5]$ and $U = [-1,1]$
respectively into $p_t = 10$, and $p_1 = p_2 = p_u = 5$ subintervals. Then the set Ω is divided into N = 1250 cells, and $y_k = (t_k, x_k, u_k)$ belongs to the kth cell, we have chosen $M_1 = 4, M_2 = 8, L = 10$, and $t_a = 1$.
In this example after 59 iterations the Optimal Time converges to the value $T^* = 1.4286$, while the exact optimal time is $\sqrt{2}$ (see Pinch, 1995). The graphs of the piecewise constant control function and trajectory are as follows (Fig. 1):



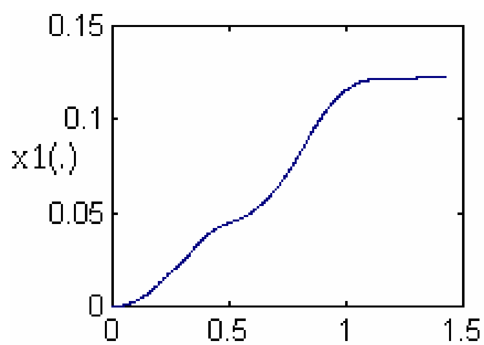


Figure 1 - Optimal Control and State

Example 2. (see Balachanran, 1999) Minimize *J*=*T* subject to the nonlinear system of equation:

$$\dot{x}_1 = x_2, \qquad x_1(T) = 0,$$
$$\dot{x}_2 = \frac{1}{2u \sec(x_2^2)}, \qquad x_2(T) = 0$$

We divided the sets $J=[0,T], A_1=[-0.5,0]$, $A_2=\left[-\sqrt{\pi/2}\ ,\ 0\right]$, $U=[-1,1]$ respectively into $p_t=5$,and $p_1=p_2=p_u=5$ subintervals. Then the set Ω is divided into $N=5^4+1$ cells, and $y_k=(t_k,x_k,u_k)$ belongs to the kth cell, and $M_1=2, M_2=8$, and $L=5$, and we consider $t_a=0.2$.

In this example after 22 iterations the Optimal Time converges to the value $T^*=0.36$.The graphs of the piecewise constant control function and trajectory are as follows (Fig. 2).

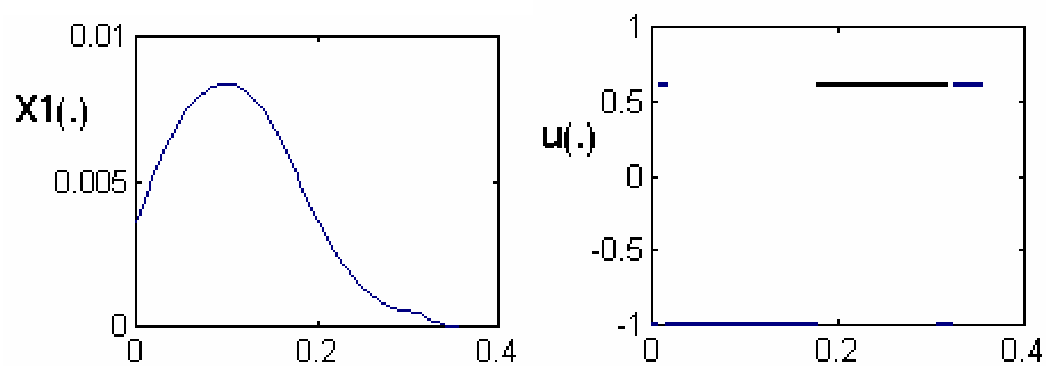


Figure 2 - Optimal Control State

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